7.1 Continue

A symmetric $n \times n$

- Has $n$ real eigenvalues (counting w/ multiplicities)
- dim eigenspace for $\lambda_i$ = algebraic multiplic. of $\lambda_i$
- Eigenspaces of different eigenvalues are orthogonal to each other
- A is orthogonally diagonalizable:

$$A = PDP^T \qquad D \text{ diagonal}$$

$$P \text{ orthogonal } PP^T = P^TP = I$$

Procedure for orthogonally diagonalize A (symmetric)

- Find eigenvalues
- Find basis for each eigenspace
- Within each eigenspace use Gram-Schmidt to ortogonalize basis.
- Orthonormalize ( $v \to \frac{v}{|v|}$ )
-

$$P = \Big[ \underbrace{v_1 \dots v_n}_{\text{o.n. basis consisting eigenvectors of } A} \Big]$$

7.1 #24 Given $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 10 \end{bmatrix} = 10\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_1 = 10.$$

$$\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 1.$$

A is 3x3

$\det A = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$ why?

~~For any~~ every $n \times n$ matrix $A$: $\det A = \lambda_1 \cdots \lambda_n$ provided that all eigenvalues in $\mathbb{C}$ (with multiplic.) are used: $\lambda_1, \dots, \lambda_n$

Reason: $\det(A - \lambda I) = \text{char. poly} = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$

plug in $\lambda = 0$ $\qquad \det A = \lambda_1 \lambda_2 \lambda_3.$

3 x 3

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \qquad a_{11} + a_{22} + \dots + a_{nn} \overset{\text{def}^n}{=} \text{trace } A$$

$\text{trace } A = \lambda_1 + \lambda_2 + \dots + \lambda_n.$

Also true for all $n \times n$ matrices $A$ provided that all eigenvalues (in $\mathbb{C}$) are used (with multiplic.) $\lambda_1, \lambda_2, \dots, \lambda_n$.

$$\det A = \begin{vmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{vmatrix} = 10 = \underset{10}{\lambda_1}\,\underset{1}{\lambda_2}\,\underset{1}{\lambda_3}$$

$$\text{Trace} = 12 = \underset{10}{\lambda_1} + \underset{1}{\lambda_2} + \underset{1.}{\lambda_3} \qquad \lambda_3 = 1$$

$\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} x_2 + \frac{1}{2} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Basis for eigenspace $\lambda = 1$

$\lambda = 10$

$$A - 10I = \begin{bmatrix} -5 & & \\ & -5 & \\ & & \end{bmatrix}$$

No need

$\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ was given.

Basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$ $u_3 \perp u_1$

$u_1 \quad u_2 \quad u_3$ $u_3 \perp u_2$.

G-S on $\{u_1, u_2\}$.

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 2 \end{bmatrix}$$

$$\text{orthogonal basis} \quad \begin{array}{ccc} \ell_1=\sqrt{2} & \ell_2=\sqrt{18} & \ell_3=3 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, & \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}. \end{array}$$

$$\text{orthonormal basis} \quad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & 4/\sqrt{18} & -\frac{2}{3} \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

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Next page for 7.2 Start

7.2 <u>Defn</u> Let $A$ be $n \times n$ symmetric matrix

Define $Q: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$Q(\vec{x}) = x^T A x$$

$Q$ is called the quadratic form associated to $A$.

p406 #2

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$Q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$= 4x_1^2 + 2x_2^2 + 1 \cdot x_3^2 + 6x_1x_2 + 0 \cdot x_1x_3 + 2x_2x_3$$

#4 $Q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 20x_1^2 + 15x_1x_2 - 10x_2^2$ in $\mathbb{R}^2$

$$A = \begin{bmatrix} 20 & 7\frac{1}{2} \\ 7\frac{1}{2} & -10 \end{bmatrix}$$

<u>Caution:</u> $Q' = 20x_1^2 + 15x_1x_2 - 10x_2^2$ in $\mathbb{R}^3$

$$A' = \begin{bmatrix} 20 & 7\frac{1}{2} & 0 \\ 7\frac{1}{2} & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

CHANGE of VARIABLES (For Quadratic Forms)

$$\vec{x} = P\vec{y} \quad \text{(Linear change of variables)}$$

$$Q(x) = x^T A x = (Py)^T A (Py) = y^T \underbrace{P^T A P}\, y$$

Recall orthogonal diagonalization for Symmetric $A$

$$PDP^T = A \qquad PP^T = I$$

$$P^{-1} = P^T$$

$$D = P^{-1} A (P^T)^{-1} \qquad (P^T)^{-1} = P.$$

$$D = P^T A P$$

If we choose $P$ to orthogonally diagonalize $A$

$$Q(x) = Q(Py) = y^T D y$$

PRINCIPAL AXIS THEOREM:

Let $A$ be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variables $x = Py$ that transforms the quadratic form $x^T A x$ into $y^T D y$ which has no $\underbrace{\text{cross-product terms.}}_{\text{Since } D \text{ is diagonal.}}$