Transpose

7.1 A^T : Writing rows of A for columns of A^T

Def^n Symmetric Matrix: $A^T = A$

$$\begin{bmatrix} 2 & -1 & 7 \\ -1 & 3 & 5 \\ 7 & 5 & 1 \end{bmatrix} \leftarrow \text{Symmetric} \qquad \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

not symmetric.

Def^n Orthogonal Matrix P, $PP^T = P^TP = I$

that is $P^{-1} = P^T$

Prop: $\underbrace{P^TP = PP^T = Id}_{n \times n \text{ matrix.}} \iff$ Columns of P form an orthonormal basis of $\mathbb{R}^n$

$\iff$ Rows of P form an orthonormal basis for $\mathbb{R}^n$

Ex

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ orthogonal matrix

- Check dot products of non-identical columns are 0

Check:
- Each column is a unit vector.

Example

$$\underbrace{\begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P^T} \underbrace{\begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{bmatrix}}_{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Both P and P^T are orthogonal.

CRUCIAL THM Let A be an $n \times n$ matrix.

A is symmetric $\Longleftrightarrow$ A is orthogonally diagonalizable

Defn $A = PDP^T$ where
- P is orthogonal $PP^T = I$
- D is diagonal

so $P^T = P^{-1}$

Why does this work:

SPECTRAL THM: Let A be a symmetric $n \times n$ matrix. Then

- A has n real eigenvalues, counting with multiplicities

For all i:
- dim of eigenspace for λ_i = multiplicity of λ_i
- Eigenspaces of different eigenvalues are orthogonal to each other
- A is orthogonally diagonalizable.

PROCEDURE:

- Given a symmetric matrix A $(n \times n)$
- Calculate $\det(A - \lambda I)$
- Find all eigenvalues: $\det(A - \lambda I) = 0$
  must have $\underline{\underline{n}}$, counting with multiplicities.
- For each eigenspace find a basis.
- Use Gram-Schmidt process to ortogonalize the basis for each eigenspace separately.
- Obtain an orthogonal basis of $\mathbb{R}^n$ by using eigenvectors from previous step.
- Normalize the basis to obtain an orthonormal basis by using eigenvectors $v_1 \dots v_n$

$$\underline{P} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

$$\underline{P}^{-1} = P^T.$$

$$A = PDP^T.$$

7.1 Exc # 14 $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ symmetric.

$$\begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25 = 1 - 2\lambda + \lambda^2 - 25$$
$$= \lambda^2 - 2\lambda - 24$$
$$= (\lambda-6)(\lambda+4)$$

$\lambda = 6, -4.$

$\lambda = 6$ $\begin{bmatrix} 1-6 & 5 \\ 5 & 1-6 \end{bmatrix} \to \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$

$x_1 = x_2$

$x_2 = $ parameter

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

basis for eigenspace for $\lambda = 6$

$\lambda = -4$ $\begin{bmatrix} 1+4 & 5 \\ 5 & 1+4 \end{bmatrix} \to \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$

$x_1 = -x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

basis for eigenspace for $\lambda = -4$

Lengths: $l_1 = \sqrt{2}$ $l_2 = \sqrt{2}$

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ orthogonal basis

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$ orthonormal "

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$A = PDP^T$$

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{P^T = P^{-1}}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ \frac{6}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{2} - \frac{4}{2} & \frac{6}{2} + \frac{4}{2} \\ \frac{6}{2} + \frac{4}{2} & \frac{6}{2} - \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}.$$

Ex $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Show orthogonally diagonalizable by finding P s.t. $A = PDP^T$.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 1 + 1 - 3(1-\lambda)$$

$$= \cancel{1} - \cancel{3\lambda} + 3\lambda^2 - \lambda^3 + \cancel{2} - \cancel{3} + \cancel{3\lambda}$$

$$= 3\lambda^2 - \lambda^3 = -\lambda^2(\lambda - 3)$$

$\lambda = 0, 0, 3$

$$A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} & \text{free} & \text{free} \end{array}$$

$x_1 + x_2 + x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Basis for eigenspace $\lambda = 0$

$$A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = x_3$

$x_2 = x_3$.

$x_3 =$ free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3.$$

Basis: (not orthogonal)

$$\begin{array}{ccc} u_1 & u_2 & u_3. \\ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \lambda = 0 & \lambda = 0 & \lambda = 3. \end{array}$$

$u_1 \perp u_3$

$u_2 \perp u_3$

$u_1 \not\perp u_2$.

$\underbrace{\lambda = 0 \quad \lambda = 0}_{\text{apply Gram Schmidt}}$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \left(\perp \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

parallel

$\ell_2 = \sqrt{6}$

$$\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{orthogonal basis for } \mathbb{R}^3 \text{ by using eigenvectors}} \leftarrow \ell_3 = \sqrt{3}$$

$\ell_1 = \sqrt{2}$

$$\overbrace{\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}}^{\text{orthonormal basis}}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$A = PDP^T$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P^T}$$