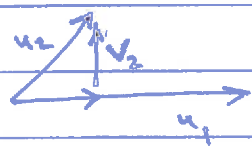
(6.4) GRAM-SCHMIDT Process:

Given a basis of a subspace W,
to obtain an orthogonal basis for W
(" " " orthonormal " " ")

dim W = 2

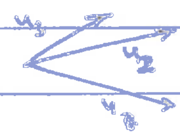


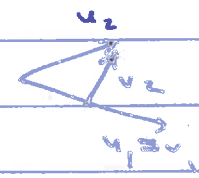
$v_1 = u_1$

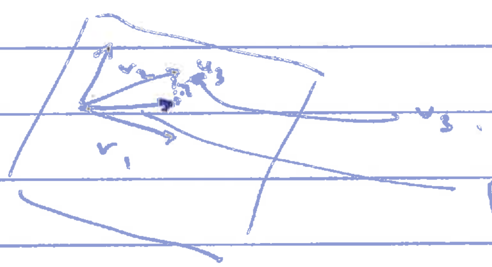
$v_2 = u_2 - \dfrac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1$

$v_1 \perp v_2$ neither 0 (if $u_1 \not\parallel u_2$)
not parallel

dim W = 3







$\text{proj}_V u_3 \longrightarrow v_3 = u_3 - \text{proj}_V u_3.$

$V = \text{Span}(v_1, v_2)$

p358 Ex #2 $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$. basis for W

- Find an orthogonal basis for W
- " " orthonormal " " W

p2

$$u_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = v_1 .$$

$$v_2 = u_2 - \frac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{0+24-14}{0+16+4} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \quad \left(\perp \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} ? \checkmark \right)$$

Orthogonal basis $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}$

length $= \sqrt{20}$
$= 2\sqrt{5}$

length
$\sqrt{25+16+64}$
$= \sqrt{105}$

Orthonormal basis $\left\{ \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{105}} \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}.$

Ex $\begin{array}{ccc} u_1 & u_2 & u_3 \\ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, & \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$ (spans $\mathbb{R}^3$)

Find any orthogonal basis of $\mathbb{R}^3$, where $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is one of the basis elements.

SOLUTION:

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$u_1 = v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} - \frac{0 + 6 - 1}{0 + 4 + 1} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}. \qquad \left(\perp \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} ? \checkmark \right)$$

$$\begin{array}{ccc} v_1 & v_2 & u_3 \\ \| & \| & \| \\ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, & \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{3}{21} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix} - \begin{bmatrix} 4/7 \\ 1/7 \\ -2/7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{4}{7} \\ 1 - \frac{6}{5} - \frac{1}{7} \\ 1 - \frac{3}{5} + \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 3/7 \\ -12/35 \\ 24/35. \end{bmatrix}$$

$$\frac{35 - 42 - 5}{35}$$

$$\frac{35 - 21 + 10}{35}$$

Theorem Gram - Schmidt :

Let W be the subspace spanned by a basis $\{u_1, u_2, \dots u_n\}$.
Then $\{v_1, v_2 \dots v_n\}$ is an orthogonal basis for W if

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_p = u_p - \sum_{i=1}^{p-1} \frac{u_p \cdot v_i}{v_i \cdot v_i} v_i$$

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{u_k \cdot v_i}{v_i \cdot v_i} v_i .$$

p359 #10

$$A = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \\ \shortparallel & \shortparallel & \shortparallel \end{array} \\ \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \end{array}.$$

Q: Find an orthogonal basis for the column space of A.

(It turns out: the columns are linearly independent. otherwise, read the next page.)

$$u_1 = v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-6-24-2-4}{1+9+1+1} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = v_2 \qquad \left(\perp \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$v_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{\cancel{-6}+9+\cancel{6}-3}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{18+3+6+3}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 + \frac{1}{2} - \frac{15}{2} \\ 3 - \frac{3}{2} - \frac{5}{2} \\ 6 - \frac{1}{2} - \frac{5}{2} \\ -3 - \frac{1}{2} + \frac{5}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Q: If a set of vectors given $\{u_1, \ldots u_p\}$ (not known to be linearly independent), and If we want an orthogonal basis for the span $\{u_1, \ldots u_p\}$; Then what do we do?

A: There is no reason to find a basis chosen among $\{u_1 \ldots u_p\}$, since Gram-Schmidt process will eliminate the dependent vector.

Simply: start with $\{u_1, \ldots u_p\}$; apply G-S.; and if any of $v_i = 0$, toss it out! Remaining non-zero v_j's will form ~~be~~ an orthogonal basis. for span $\{u_1, \ldots u_p\}$.