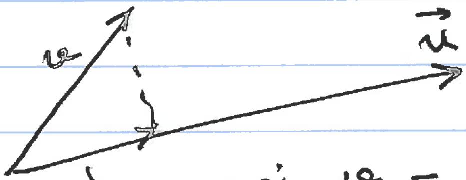
6.2 Review



$$\text{proj}_u v = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

- If $\{\vec{u}_1, \dots \vec{u}_p\}$ is an $\underbrace{\text{orthogonal}}$ basis for W,

  $u_i \cdot u_j = 0 \quad i \neq j$

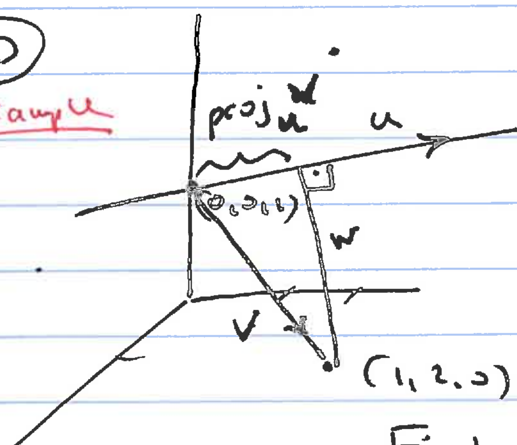
  $\|u_i\|^2 = u_i \cdot u_i \neq 0$ all i,j

  $w \in W$ then

$$w = \sum_{i=1}^{p} \frac{w \cdot u_i}{u_i \cdot u_i} u_i$$

3D

Example



L: $\vec{x}(t) = (0,0,1) + t\,(1,2,-2)$

Line thru $(0,0,1)$

& parallel to $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \vec{u}$

Find distance from $(1,2,0)$ to L.

Sol'n:

$$\text{proj}_u v = \frac{v \cdot u}{u \cdot u} u =$$

$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = u =$ direction of the line $\qquad v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

p2

$$\mathrm{proj}_u v = \frac{v \cdot u}{u \cdot u} u = \frac{\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-2\end{bmatrix}}{\begin{bmatrix}1\\2\\-2\end{bmatrix} \begin{bmatrix}1\\2\\-2\end{bmatrix}} \begin{bmatrix}1\\2\\-2\end{bmatrix}$$

CORRECTION of Algebra:

$$= \frac{1+4+2}{1+4+4} \begin{bmatrix}1\\2\\-2\end{bmatrix}$$

$$= \frac{7}{9} \begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}7/9\\14/9\\-14/9\end{bmatrix} \qquad \cancel{= \frac{3}{9} \begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix}}$$

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}7/9\\14/9\\-14/9\end{bmatrix} = \begin{bmatrix}2/9\\4/9\\5/9\end{bmatrix} = w = v - \mathrm{proj}_u v = \cancel{\begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\-1/3\end{bmatrix}} =$$

$$\sqrt{\frac{4}{81} + \frac{16}{81} + \frac{25}{81}} = \mathrm{dist}(P, L) = \|w\| = \cancel{\sqrt{\frac{4}{9} + \frac{16}{9} + \frac{1}{9}} = \frac{\sqrt{21}}{3}}$$

$$= \frac{\sqrt{5}}{3}$$

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u & v Orthogonal $\iff$ $u \cdot v = 0$

Orthogonal set $\{\vec{u}_1, \dots \vec{u}_p\}$ $\iff$ $\left\{ \begin{array}{l} \text{All } i, j \\ u_i \cdot u_j = 0 \\ \text{if } i \neq j \end{array} \right.$

Orthogonal basis $\{\vec{u}_1, \dots \vec{u}_p\}$ $\iff$ $\left\{ \begin{array}{l} \text{All } i, j \\ u_i \cdot u_j = 0 \\ u_i \cdot u_i \neq 0 \end{array} \right.$

Defn Orthonormal Set : $\{\vec{u}_1, \vec{u}_2 \dots \vec{u}_p\}$

- Orthogonal set, and
- $\|u_i\| = 1$ all i

Defn Orthonormal basis : Basis + orthormal p345 **Ex 20:**

$$\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

Is it orthogonal? YES!

$$\left(\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}\right) = -\frac{2}{9} + \frac{2}{9} + 0 = 0$$

Is it orthonormal.

$$\left\| \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1 \checkmark$$

$$\left\| \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{9} + \frac{4}{9}} = \frac{\sqrt{5}}{3} \neq 1$$

Not orthonormal.

Orthonormal $\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \frac{3}{\sqrt{5}} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\}.$

Thm: If U is an $m \times n$ matrix which has orthonormal columns then

$$U^T U = I$$

Ex

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

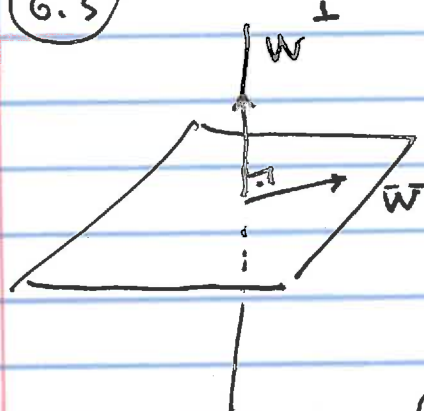
Caution ORDER!

det = 0

$$\overbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}^{U^T} = \underbrace{\begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}}_{3 \times 3} \neq I$$

6.2 ✓

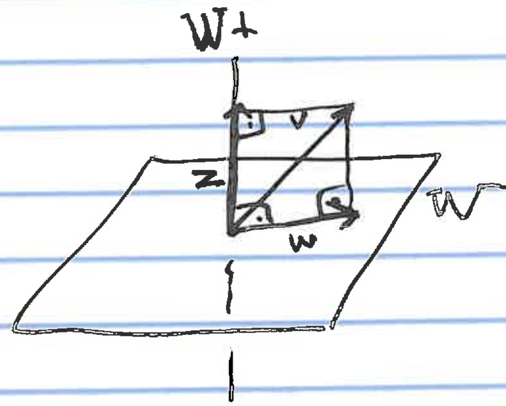
6.3



Defn Let W be a non-empty subspace of $\mathbb{R}^n$.

$$W^\perp = \{ \vec{z} \in \mathbb{R}^n \mid z \cdot w = 0 \text{ for all } w \in W \}$$

Called orthogonal Complement of W



$$v = \underset{W}{\underset{\cap}{w}} + \underset{W^\perp}{\underset{\cap}{z}}$$

Thm: Let W be a non-empty subspace of $\mathbb{R}^n$.

For any $\vec{v} \in \mathbb{R}^n$, one can write

$$\vec{v} = \vec{w} + \vec{z} \quad \text{where}$$

$$w \in W, \quad z \in W^\perp.$$

If $\{u_1, \dots, u_p\}$ is an orthogonal basis for W,

then $$\vec{w} = \sum_{i=1}^{p} \frac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i} \vec{u}_i, \qquad \vec{z} = \vec{v} - \vec{w}$$

Ex 1 $\quad y = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}, \quad \underbrace{u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{orthogonal set}}$

$W = \text{Span}\,\{\vec{u}_1, \vec{u}_2\}$

want to write $\quad y = \underset{\substack{\Uparrow \\ W}}{w} + \underset{\substack{\Uparrow \\ W^{\perp}}}{z}$.

$$w = \frac{u_1 \cdot y}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} u_2$$

$$= \frac{\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{-28 - 2 + 3}{16 + 1 + 1} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{0 + 2 + 3}{0 + 1 + 1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{-27}{18} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = w \in W = \text{Span}\,\{u_1, u_2\}.$$

Ex 1 Continue

$$\vec{y} = \vec{w} + \vec{z}$$

$$\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} + \vec{z} \qquad z = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

(Check! better $\perp u_1, u_2$)

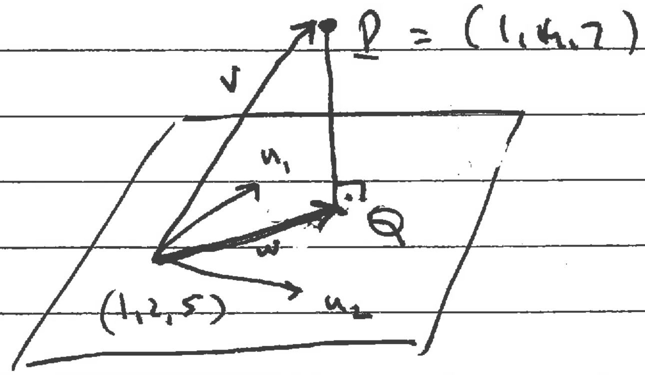
$$\left.\begin{array}{l} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} = -4 + 2 + 2 = 0 \\ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 - 2 + 2 = 0. \end{array}\right\} \text{ok} \checkmark$$

go to next page!

Ex 2 Find the closest pt Q of the plane

$$(1,2,5) + t\,\underbrace{(1,2,-1)}_{u_1} + s\,\underbrace{(2,0,2)}_{u_2}$$

to the point $P = (1,4,7)$.



Check!

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}2\\0\\2\end{bmatrix} = 0$$

orthogonal ✓

then

Let $W = \text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$

$$v = \begin{bmatrix}1\\4\\7\end{bmatrix} - \begin{bmatrix}1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}$$

$$\text{proj}_W v = w = \frac{\begin{bmatrix}0\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}}{\begin{bmatrix}1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\-1\end{bmatrix}} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{\begin{bmatrix}0\\2\\2\end{bmatrix} \cdot \begin{bmatrix}2\\0\\2\end{bmatrix}}{\begin{bmatrix}2\\0\\2\end{bmatrix} \cdot \begin{bmatrix}2\\0\\2\end{bmatrix}} \begin{bmatrix}2\\0\\2\end{bmatrix}$$

Lower case w

$$= \frac{0+4-2}{1+4+1} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{0+0+4}{4+0+4} \begin{bmatrix}2\\0\\2\end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{1}{2} \begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}4/3\\2/3\\2/3\end{bmatrix} = w.$$

$$Q = \begin{bmatrix}1\\2\\5\end{bmatrix} + \begin{bmatrix}4/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}7/3\\8/3\\17/3\end{bmatrix}.$$

HW

6.3 # 1, 3, 5, 11