4/21/14

(p1)

6.2

Def $u \cdot v = 0 \iff$ "u is orthogonal to v"
"u and v are orthogonal"

Def A set S is called orthogonal,
if each pair of distinct vectors $\vec{u}$ and $\vec{v}$
one has: $\vec{u} \cdot \vec{v} = 0$.
That is: $S = \{\vec{u}_1, \vec{u}_2, \dots \vec{u}_p\}$ is orthogonal,
if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

Ex $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set. Since:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0 - 2 + 2 = 0$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix} = -5 + 1 + 4 = 0$$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix} = 0 - 2 + 2 = 0$$

Prop Given a set of non-zero vectors: $S = \{\vec{u}_1, \dots \vec{u}_p\}$.
if S is an orthogonal set, then it is
linearly independent.
Hence S is a basis for its span.

Defn An orthogonal basis for a subspace W is
- a basis for W and
- an orthogonal set.

Ex $(2,5,-1) = \underbrace{c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)}_{\text{Find } c_1, c_2, c_3}$

$c_1 = 2$

$c_2 = 5$ ← easy

$c_3 = -1.$

Thm: If $S = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_p\}$ is an orthogonal basis for W, then for all $v \in W$

$$v = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p \quad \text{where}$$

$$c_i = \frac{\vec{v} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$$

p345 **Ex** #9 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

- Check orthogonal basis.

Find c_1, c_2, c_3 • $\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$

$$\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\1\end{bmatrix} = -1+0+1=0$$

$$\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = 2+0-2=0$$

$$\begin{bmatrix}-1\\4\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = -2+4-2=0$$

- orthogonal ✓
- linearly indep. since all non-zero vectors
- Span $\mathbb{R}^3$, since 3 linearly independent vectors automatically span $\mathbb{R}^3$ (since both Span$\{u_1, u_2, u_3\}$ & $\mathbb{R}^3$ are 3 dimensional)

$$\begin{bmatrix}8\\-4\\-3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\4\\1\end{bmatrix} + c_3\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

$$c_1 = \frac{\begin{bmatrix}8\\-4\\-3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}} = \frac{8+0-3}{1+0+1} = \frac{5}{2}$$

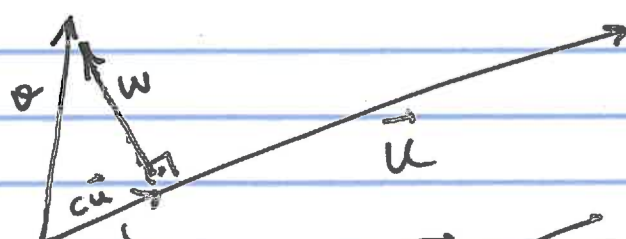
$$c_2 = \frac{\begin{bmatrix}8\\-4\\-3\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\1\end{bmatrix}}{\begin{bmatrix}-1\\4\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\1\end{bmatrix}} = \frac{-8-16-3}{1+16+1} = \frac{-27}{18} = -\frac{3}{2}.$$

$$c_3 = \begin{bmatrix}8\\-4\\-3\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix}\Big/\begin{bmatrix}2\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = \frac{16-4+6}{4+1+4} = \frac{18}{9} = 2$$

$$\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \stackrel{?}{=} \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} + \frac{3}{2} + 4 \\ 0 - 6 + 2 \\ \frac{5}{2} - \frac{3}{2} - 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} \checkmark$$

ORTHOGONAL PROJECTION



$\text{proj}_{\vec{u}} \vec{v}$ → • parallel to u
→ $w \perp u$

$$\text{proj}_{\vec{u}} \vec{v} = c \cdot \vec{u} \qquad c \in \mathbb{R}.$$

$$cu + w = v$$

$$\vec{u} \cdot (c\vec{u} + \vec{w}) = \vec{u} \cdot \vec{v}$$

$$c \cdot \vec{u} \cdot \vec{u} + \underbrace{\vec{u} \cdot \vec{w}}_{0} = \vec{u} \cdot \vec{v}$$

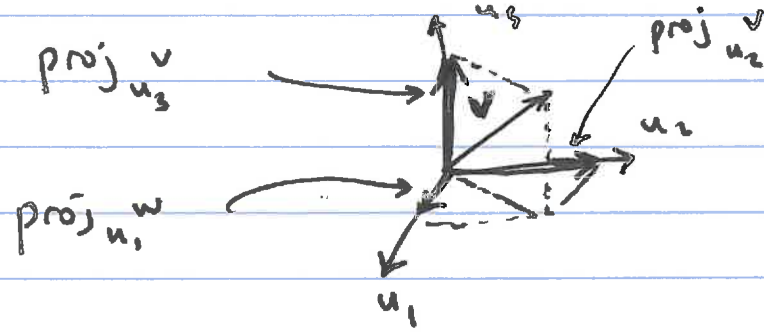
$$c(\vec{u} \cdot \vec{u}) = \vec{u} \cdot \vec{v}$$

$$c = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}.$$

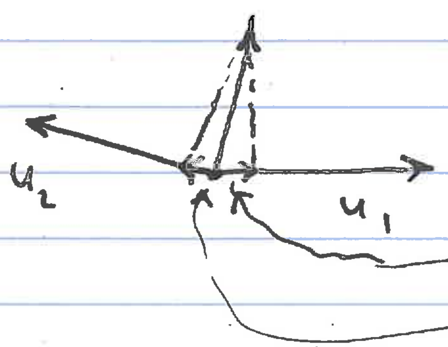
Go back to Thm 5:

$\{\vec{u}_1, \dots \vec{u}_p\}$ orthogonal basis

$$\boxed{\vec{V} = \sum_{i=1}^{p} u_i \frac{\vec{V}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}}$$



False: if $\{u_1, u_2, \dots u_p\}$ is not orthogonal



$v \neq \text{proj}_{u_1} v + \text{proj}_{u_2} v$

Exc #14 $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Want : $y = c \cdot \vec{u} + \vec{w}$

$u \perp w$

$$c = \frac{y \cdot u}{|u|^2} = \frac{\begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}}{50} = \frac{14 + 6}{50} = \frac{2}{5}$$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + w$$

$$w = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$$

$w \perp u$??

$$\begin{bmatrix} 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix} = -\frac{28}{5} + \frac{28}{5} = 0. \checkmark$$