6.1 Lots of new vocabulary!

Def

I Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$.

Then one defines inner product (dot product) of $\vec{u}$ and $\vec{v}$ to be

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

Obs If one looks at $\vec{u}$ and $\vec{v}$ as $n \times 1$ matrices then

$$[u \cdot v] = u^T \cdot v$$

Ex $$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ 6 \end{bmatrix} = -21 + 30 = 9.$$

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = 6 - 3 + 20 = 23$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix} \quad \text{not defined}$$

$$\begin{bmatrix} 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 23 \end{bmatrix}$$

<u>Properties of $\cdot$</u> For $u, v, w \in \mathbb{R}^n$

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

For $c \in \mathbb{R}$

- $(c\vec{v})\cdot\vec{w} = c\cdot(\vec{v}\cdot\vec{w})$
- $\vec{u}\cdot\vec{u} \geq 0 \quad (\vec{u}\cdot\vec{u} = 0 \iff \vec{u} = \vec{0})$

Ex. $\begin{bmatrix}5\\4\\-1\end{bmatrix}\cdot\begin{bmatrix}5\\4\\-1\end{bmatrix} = 5^2 + 4^2 + (-1)^2 = 42$

KNOW:

<u>Question</u>. $A, B, C$ matrices $(A\cdot B)\cdot C = A\cdot(B\cdot C)$ provided that matrix multiplication is possible (TRUE)

Is it true that

$(\vec{u}\cdot\vec{v})\cdot\vec{w} = u\cdot(\vec{v}\cdot\vec{w})$ ?

False:

$$\Big(\underbrace{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}}_{5}\Big)\cdot\begin{bmatrix}1\\2\end{bmatrix} \neq \begin{bmatrix}1\\1\end{bmatrix}\cdot\Big(\underbrace{\begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}}_{8}\Big)$$

$$\begin{bmatrix}5\\10\end{bmatrix} \neq \begin{bmatrix}8\\8\end{bmatrix}$$

Different "$\cdot$"s

products

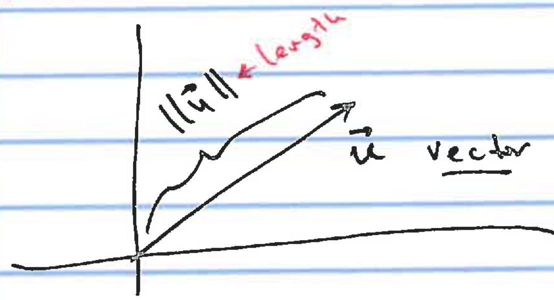
**LENGTH:** if $\vec{u} \in \mathbb{R}^n$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Then we defines

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}$$

$$= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

so that $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$




$\vec{u}$ vector showing displacement

Ex. $$\left\| \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \right\| = \sqrt{4+9+25} = \sqrt{38}$$

A Unit vector is a vector of length 1.

$$\left\| \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ -1/2 \end{bmatrix} \right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}} = 1.$$

**Normalization:** $V \longrightarrow \dfrac{V}{\|V\|}$ if $\vec{v} \neq 0$

This gives a vector that is unit and in the same direction as the original.

Ex $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ normalize (that is, to find a vector that is unit in the same direction.

$$\left\| \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\frac{\vec{v}}{\|v\|} = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \longrightarrow \text{unit}$$

Q: Is it true $\left\| \frac{v}{\|v\|} \right\| = 1$ always ? YES:

Proof: $\left\| \frac{v}{\|v\|} \right\|^2 = \frac{v}{\|v\|} \cdot \frac{v}{\|v\|} = \frac{v \cdot v}{\|v\|^2} = \frac{\|v\|^2}{\|v\|^2} = 1.$

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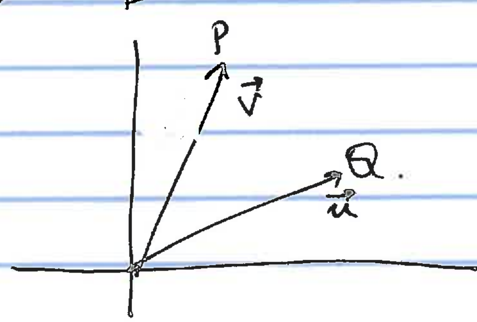
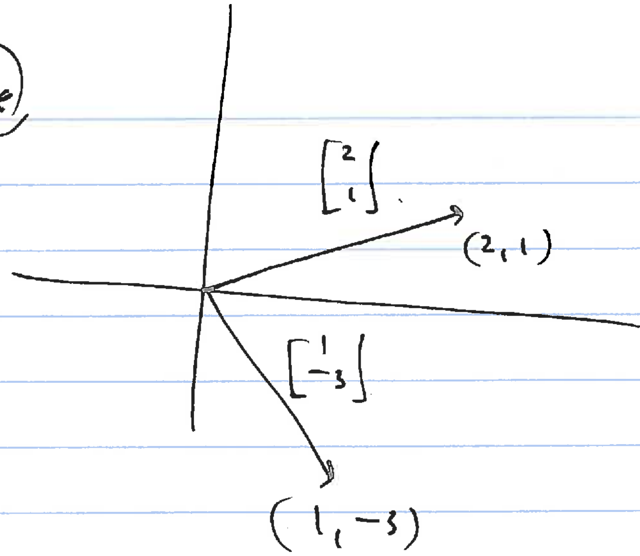
## II Distance




Distance between $\vec{u}$ and $\vec{v}$, actually means the distance between the terminal pts.

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|\vec{v} - \vec{u}\|$$

Ex




$$\text{dist}\left(\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\-3\end{bmatrix}\right)=\sqrt{(2-1)^2+(1-(-3))^2}$$

$$=\sqrt{1+16}=\sqrt{17}.$$

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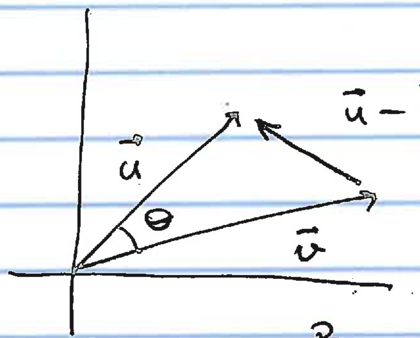
## Discussion




Linear Alg.

$$\left\{\begin{aligned} \|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= u\cdot u - u\cdot v - v\cdot u + v\cdot v \\ &= \|u\|^2 + \|v\|^2 - 2\cdot(u\cdot v) \end{aligned}\right.$$

Law of cosines

Compare terms.

$$c^2 = \|u-v\|^2 = a^2 + b^2 - 2ab\cos\theta$$
$$= \|u\|^2 + \|v\|^2 - 2\|v\|\,\|u\|\cos\theta.$$

To Conclude: $\|u\|\,\|v\|\cos\theta = u\cdot v$

$$\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|} \quad \text{if } \vec{u}\neq 0,\ \vec{v}\neq 0$$

This formula extends to $\mathbb{R}^n$ since the triangle with vertices $\vec{0}, \vec{u}, \vec{v}$ is contained in a 2-plane, which is metrically the same as $\mathbb{R}^2$.

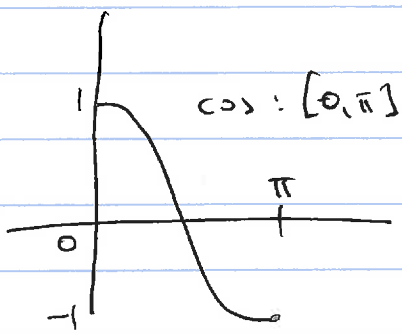
## Consequently

**Defn** $\vec{u}\cdot\vec{v} = 0 \iff u$ and $v$ are called <u>orthogonal</u> to each other.

( i.e. $u \perp v \quad \theta = \pi/2$ )

**Defn** Angle between non-zero vectors $u, v$

$$\theta = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\,\|v\|}\right) \qquad 0 \le \theta \le \pi$$




$\cos : [0,\pi] \longrightarrow [-1,1]$.

$\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$.

Ex $\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}-1\\-5\\2\end{bmatrix}$ are they orthogonal?

$\begin{bmatrix}3\\1\\4\end{bmatrix}\cdot\begin{bmatrix}-1\\-5\\2\end{bmatrix} = -3-5+8 = 0$ Yes!

Ex Find the angle between $\begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}-3\sqrt{5}\\-\sqrt{7}\\2\sqrt{7}\end{bmatrix}$.

$\left\|\begin{bmatrix}2\\2\\1\end{bmatrix}\right\| = \sqrt{4+4+1} = \sqrt{9} = 3.$

$$\left\| \begin{bmatrix} -3\sqrt{5} \\ -\sqrt{7} \\ +2\sqrt{7} \end{bmatrix} \right\| = \sqrt{45 + 7 + 28} = \sqrt{80}$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3\sqrt{5} \\ -\sqrt{7} \\ 2\sqrt{7} \end{bmatrix} = -6\sqrt{5} \cancel{-2\sqrt{7}} \cancel{+2\sqrt{7}}$$

$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|} = \frac{-6\sqrt{5}}{3 \cdot \sqrt{80}} = \frac{-6\cancel{\sqrt{5}}}{3 \cdot 4\cancel{\sqrt{5}}} = -\frac{1}{2}.$$

$$\theta = \frac{2\pi}{3} \quad (120^\circ)$$

SKIP: ORTHOGONAL COMPLEMENTS
middle of p334 — middle of 335 including Th 3

we do/did Angles in $\mathbb{R}^2, \mathbb{R}^3$.