4/9/14

p1

p 286 Exc #8

$$\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$$ Is it diagonalizable?

$$\begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 0 \implies \lambda = 3, 3$$

$$A - 3I = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

param. pivot

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & ? \\ 0 & ? \end{bmatrix}$$ No such P

A is <u>NOT</u> diagonalizable since A is 2×2 & we cannot find a basis of two vectors, where each is an eigenvector corresponding to $\lambda = 3$

p 286 #12 $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ Find P, D s.t.

$\left.\begin{array}{l} A = PDP^{-1} \\ P \text{ invertible} \\ D \text{ diagonal} \end{array}\right\}$ if possible!

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}_{-C_3+C_2} = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & -2+\lambda & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

$R_2 + R_3$

$$= (2-\lambda) \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((3-\lambda)(4-\lambda) - 2 \right)$$

$$= (2-\lambda)\left(12 - 7\lambda + \lambda^2 - 2\right) = (2-\lambda)(\underbrace{\lambda^2 - 7\lambda + 10}_{(\lambda-5)(\lambda-2)})$$

$\lambda = 2, 2, 5.$

Find basis for eigenspaces

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} & \uparrow & \uparrow \\ & x_2 & x_3 \end{array} \end{array}$$

$x_1 + x_2 + x_3 = 0$

x_2, x_3 parameters

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Basis for eigenspace $\lambda = 2$

$\lambda = 5$

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \cdots \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ RREF}$$

$\uparrow$ x_3 parameter

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

basis for eigenspace $\lambda = 5$

$$P = \begin{array}{c} \begin{bmatrix} -1 & -1 & 1 \\ +1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \lambda=2 & \lambda=2 & \lambda=5 \end{array} \end{array} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

#12 continue

To check $A = PDP^{-1}$

$\Leftrightarrow AP = PD$ if P is invertible

① check!

$$\det P = \begin{vmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= -1(-2) - (-1)$$

$$= 2 + 1 = 3. \neq 0$$

ok

P is invertible

② $AP = PD$ Check!

$$PD = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\parallel$$

$$AP = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{bmatrix}$$

✓

#16 $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 5-\lambda & -2 \\ 1 & 3 & 1-\lambda \end{vmatrix} = -\lambda\left(\lambda^2 - 6\lambda + 16\right)$$

$R_2 = R_1 + R_3 \Longrightarrow \det A = 0$ since $RREF = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{bmatrix}$

$\det(A - \lambda I) = 0$ holds when $\lambda = 0$

$\lambda^2 - 6\lambda + 16 = 0$ has no real roots!

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\qquad b^2 - 4ac = 36 - 4(16) < 0$

$\Downarrow$

Char poly has complex roots

$\Downarrow$

Not diagonalizable over $\mathbb{R}$

Prop If an $n \times n$ matrix has distinct n real eigenvalues, then it is diagonalizable.

Diagonalization

Summarize: Let A be $n \times n$.

- Find all real eigenvalues by solving $\det(A-\lambda I)=0$.
- List the real eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$
  (Do not repeat them in this list).

- Let m_i be the multiplicity of λ_i in the characteristic polynomial. This is called algebraic multiplicity. For $i=1,2,\ldots k$.
- Let d_i be the dimension of the eigenspace for λ_i. That is the number of basis vectors in the eigenspace for λ_i, obtained by $A-\lambda_i I \to$ RREF.

List of distinct Real Eigenvalues	Algebraic Multiplicity		Dimension of the eigenspace	
λ_1 :	m_1	$\geq$	d_1	≥ 1
λ_2 :	m_2	$\geq$	d_2	≥ 1
$\vdots$	$\vdots$		$\vdots$	
λ_i :	m_i	$\geq$	d_i	≥ 1
$\vdots$			$\vdots$	
λ_k :	m_k	$\geq$	d_k	≥ 1

Always true

Thm: A is diagonalizable $\iff d_1+d_2+\cdots+d_k=n$.

$$\iff \begin{cases} m_1+m_2+\cdots m_k = n \quad \text{and} \\ m_i = d_i \text{ for all } i=1,\ldots k. \end{cases}$$

Corollaries: (A is $n \times n$)

① n distinct real eigenvalues $\Rightarrow$ A is diagonalizable

② $m_1+m_2+\cdots+m_k < n$ (non-real complex roots) $\Rightarrow$ Not diagonalizable in $\mathbb{R}$.

③ For any i, $m_i > d_i$ $\Rightarrow$ Not diagonalizable.

#18 $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$ $\underbrace{\lambda = 0, -1, -2}_{\text{given}}$ 3 distinct eiv. for a 3×3 matrix.

$$\text{Ch poly} = (0-\lambda)(-1-\lambda)(-2-\lambda)$$

$\downarrow$
degree=3

$m_1 = m_2 = m_3 = 1.$

$$A - 0\cdot I = \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \longrightarrow \cdots \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$A + I = \begin{bmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3/2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}.$$

$$A + 2I = \begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{array}{ccc} \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & \tfrac{1}{2} & 1 \\ 1 & 1 & 0 \end{bmatrix}} \\ \uparrow & \uparrow & \uparrow \\ \lambda=-2 & \lambda=-1 & \lambda=0 \end{array} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$