5.3 DIAGONALIZATION

Defn A square matrix A is called diagonalizable if it is similar to a diagonal matrix: that is: there is an invertible $n \times n$ matrix P and a diagonal matrix D s.t

$$\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \longleftarrow$$

$$A = PDP^{-1}$$

STANDARD TRICK $\quad A = PDP^{-1}$, find A^k.

$$A^{100} = (PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$$
$$= P D \cancel{P^{-1} P} D \cancel{P^{-1} P} D \cancel{P^{-1} P} \cdots \cancel{P} D P^{-1}$$
$$= P \cdot D^{100} \cdot P^{-1}$$

Ex #2 p286 $\quad P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

$A = PDP^{-1}$ want A^4

$$A^4 = PD^4P^{-1} \qquad D^4 = \begin{bmatrix} 1^4 & 0 \\ 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 162 \\ 2 & 243 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 321 & -160 \\ 480 & -239 \end{bmatrix}$$

$$\longrightarrow P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad P^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

Caution

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} \neq \begin{bmatrix} 2^2 & 3^2 \\ 4^2 & 5^2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^3 = \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 19 \\ 0 & 27 \end{bmatrix}$$

p286 Exc 4

Find $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = ?$

Given $$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$A \quad = \quad P \quad D \quad P^{-1}$$

$$A^k = P \cdot D^k P^{-1}$$

$$A^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-3)^k & (-2)^{k+1} \\ 2(-3)^k & -(-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-3)^{k+1} + (-2)^{k+2} & 6(-3)^k + 3(-2)^{k+1} \\ -2(-3)^k - (-2)^{k+1} & 4(-3)^k + 3(-2)^k \end{bmatrix}.$$

Recall **Defn** A is diagonalizable

$\iff A = PDP^{-1}$ for some invertible P
for some diagonal D.

- How do I find P and D?
- Can I always find it? NO : Sometimes possible
& sometimes not possible

Thm A Let A be an $n \times n$ matrix.

A is diagonalizable $\overset{\text{book}}{\iff}$ A has n linearly independent eigenvectors.

$\Updownarrow$ my wording $\Updownarrow$

There is a basis of $\mathbb{R}^n$ consisting n eigenvectors of A. $\iff$ $A = PDP^{-1}$ (P invertible, D diagonal)

B $\left.\begin{array}{l} A = PDP^{-1} \\ \text{with } P \text{ invertible} \\ D \text{ diagonal} \end{array}\right\} \iff$

- columns of P are n linearly independent eigenvector of A.
- Diagonal entries of D are the eigenvalues of A (all + with multiplicity)
- k th column of P is an eigenvector of A corresponding ~~a~~ k^{th} eigenvalue of A i.e $d_{kk} = \lambda_k$

all

* Caution: You can order the basis of eigenvectors in any order, but you must use the same order for eigenvalues with multiplicity

Ex 10 $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-12$$
$$= 2 - 3\lambda + \lambda^2 - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda-5)(\lambda+2)$$

$\lambda = -2, 5$

$\lambda = -2$ $\quad A + 2I = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

y parameter

$x + y = 0$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ → basis for eigenspace $-(-2)$

$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 5$ $\quad A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}$ $\quad x - \frac{3}{4}y = 0$

parameter y

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/4\, y \\ y \end{bmatrix} = y\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ → basis for e.space $-(\lambda=5)$

$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}$ $\qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ $\qquad$ Watch the order.

$$A = PDP^{-1}$$

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

$$\text{det} = -7 \qquad P^{-1} = -\frac{1}{7} \begin{bmatrix} 4 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$