4/2/14 (p1)

5.2

Do not read the bottom half p 274 + Example 2 of p 275.

<u>VERY IMPORTANT</u>:

λ is an eigenvalue of a $n \times n$ matrix A

$\Longleftrightarrow$ The equation $A \cdot \vec{v} = \lambda \vec{v}$ has a $\vec{v} \neq 0$ sol^n.

$\Longleftrightarrow$ " " $(A - \lambda I)\vec{v} = 0$ " " $v \neq 0$ sol^n.

By Thm 8. of 2.3

$\Longleftrightarrow$ $A - \lambda I$ is not invertible

$\Longleftrightarrow$ $\det(A - \lambda I) = 0$.

<u>Defn</u> $\det(A - \lambda I) = 0$ is called the characteristic equation of A.

- Actually it is a polynomial of degree n, if A is $n \times n$.
- The eigenvalues of A are the roots of the characteristic polynomial, and vice versa.

Exc #2 p 279

$$A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{vmatrix} = (-4-\lambda)(1-\lambda) - (-6)$$

$$= -4 + 4\lambda - \lambda + \lambda^2 + 6 = \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2)$$

$\lambda = -1, -2$ eigenvalues.

$\lambda = -1$ $$A - \lambda I = A + I = \begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

y parameter

$$x + \tfrac{1}{3}y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

basis for eigenspace for $\lambda = -1$

Check: $$\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$\lambda = -2$ $$A - \lambda I = A + 2I = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

y

$$x + \tfrac{1}{2}y = 0 \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

basis for eigenspace for $\lambda = -2$

Check: $$\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Exc. #6 $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$ find eigenvalues

$$\begin{vmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = (9-\lambda)(5-\lambda)+4$$

$$= 45 - 9\lambda - 5\lambda + \lambda^2 + 4$$
$$= \lambda^2 - 14\lambda + 49$$
$$= (\lambda - 7)^2.$$

$$\lambda = \underbrace{7, 7}_{\text{multiplicity } 2.}$$

Exc. #10 $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$ find eigenvalues & char. polynomial.

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{vmatrix}$$

$$= (5-\lambda)\big((3-\lambda)(7-\lambda)+2\big)$$

$$= (5-\lambda)(21 - 3\lambda - 7\lambda + \lambda^2 + 2)$$
$$= (5-\lambda)(\underbrace{\lambda^2 - 10\lambda + 23})$$

$$\lambda = \frac{10 \pm \sqrt{100-92}}{2} = \frac{10 \pm \sqrt{8}}{2}$$

$\lambda = 5$ —— OR —— $\lambda = 5 \pm \sqrt{2}$

Ex $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 7 & 6 & 1 & 0 \\ -1 & 7 & 4 & 3 \end{bmatrix}$ e.v. 2, 2, 1, 3

ch poly. $(2-\lambda)^2(1-\lambda)(3-\lambda)$

Ex $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -3 & 3 & 2 \end{bmatrix}$ Find char. poly + eigenvalues.

(a) $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 0-\lambda & 1 \\ -3 & 3 & 2-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & 1 \\ 2 & 0-\lambda \\ -3 & 3 \end{matrix}$

$= \left((1-\lambda)(0-\lambda)(2-\lambda) + 6 - 3\right) - \left(3\lambda + 3 - 3\lambda + 4 - 2\lambda\right)$

$= \ldots = -\lambda^3 + 3\lambda^2 - 4.$ How do we find roots?

Search for rational roots $\rightarrow \frac{p}{q}$ p divides -4, q divides -1 possible rational roots

$\pm 4, \pm 2, \pm 1$

plug them into $p(\lambda)$
see which ones are roots.

$p(\lambda) = -\lambda^3 + 3\lambda^2 - 4$

Lucky guess $p(-1) = 1 + 3 - 4 = 0.$ by Long division

$\Rightarrow p(\lambda) = (\lambda+1)(\lambda^2 - 4\lambda + 4)$

$= (\lambda+1)(\lambda-2)^2$

PTo for easier way

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -3 & 3 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & -\lambda & 1 \\ -3 & 3 & 2-\lambda \end{vmatrix} \underset{C_2-C_3}{=} \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -\lambda-1 & 1 \\ -3 & 1+\lambda & 2-\lambda \end{vmatrix}$$

$$= (1+\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1 & 1 \\ -3 & 1 & 2-\lambda \end{vmatrix}$$

R_2+R_3

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix}$$

$$= -(1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-4\lambda+4)$$

$\lambda = 1$ $\qquad$ $\lambda = \underbrace{2,\ 2}_{\text{double root.}}$

F. Th. of Algebra

If $p(\lambda)$ is a polynomial of degree $n \geq 1$, with coeff in $\mathbb{C}$, then $p(\lambda)$ has exactly n roots within $\mathbb{C}$, provided that the roots are counted with multiplicity.

Ex. $\lambda^4 - 1 = 0$

$$(\lambda - 1)(\underbrace{\lambda^3 + \lambda^2 + \lambda}_{} + 1) = 0$$

$$(\lambda-1)(\lambda+1)(\lambda^2+1) = 0$$

$$\underbrace{\lambda = 1 \qquad \lambda = -1 \qquad \lambda = \pm i}_{}$$

4 roots in $\mathbb{C}$, but 2 roots in $\mathbb{R}$.

||

degree of $\lambda^4 - 1$.

SIMILAR MATRICES

Defn Two $n \times n$ matrices A and B are called similar if there exists an invertible matrix P ($n \times n$) s.t.

$$A = PBP^{-1}.$$

Obs ① If A is similar to B then $\det A = \det B$.

Why?
$$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1})$$

reciprocals

$$= \det B.$$

Obs ② If A is similar to B then
$$\det(A - \lambda I) = \det(B - \lambda I)$$
i.e. they have the same characteristic poly.

$$A = PBP^{-1} \quad \text{given}$$

$$\begin{aligned} P(B - \lambda I)P^{-1} &= PBP^{-1} - P(\lambda I)P^{-1} \\ &= A - \lambda PIP^{-1} \\ &= A - \lambda PP^{-1} \\ &= A - \lambda I \end{aligned}$$

$$\det(A-\lambda I) = \det\left(P\,(B-\lambda I)\,P^{-1}\right)$$

$$= \underbrace{\det P \cdot \det(B-\lambda I) \cdot \det P^{-1}}_{\text{reciprocals}}$$

$$= \det(B-\lambda I)$$

Conclusion: if $A = PBP^{-1}$ then

A & B have the same eigenvalues
(same multiplicities).