3/31/14

(p1)

5.1

A $n \times n$ matrix:

Want $A \cdot \vec{v} = \lambda \vec{v}$ ← vector $\vec{v} \neq 0$

A: $n \times n$; $\vec{v}$: vector; λ: real #

$\vec{v}$ ↑ eigenvector

λ called eigenvalue

Ex

p271 #4 $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} = A \qquad v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Is v an eigenvector of A?

Sol^n

$$\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Yes, there is such $\lambda = 3$;
and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A
for the eigenvalue 3.

p271 Exc #6 $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

Is v an eigenvector of A? NO, because

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \overset{?}{\neq} \lambda \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

No such λ!

IMPORTANT

want $v \neq 0$

want $A\vec{v} = \lambda\vec{v} \rightarrow A \cdot v = \lambda I v$

$\rightarrow Av - \lambda I v = 0$

$\rightarrow (A - \lambda I) \cdot v = 0 \quad (*)$

eigenvalue/vector eq.

Finding a solution (non-trivial) of a homogenous system.

Recall Th 8 from 2.3 p.112

(d) $B\vec{x} = \vec{0}$ has only trivial solution
(a) $\Longleftrightarrow$ B is invertible

} If B is $n \times n$.

Conclusion for $(*)$ $A - \lambda I$ must not be invertible

$\Longrightarrow \det(A - \lambda I) = 0.$

p272 #12 $\quad A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$ $\quad$ given $\lambda = 3, 7$ eigenvalues

Find basis for each eigenspace.

Soln Does there exist $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ s.t.

$$\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$ Yes

given already

But want $\begin{bmatrix} x \\ y \end{bmatrix}$.

last week

$$\begin{cases} 4x + y = 3x \\ 3x + 6y = 3y \end{cases} \longrightarrow \begin{matrix} x + y = 0 \\ 3x + 3y = 0. \end{matrix}$$

This week

$$\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\left(\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Compare!

short cut we must learn.

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$ $\quad x + y = 0$, y free $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

$\uparrow$ y = free

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix} \right\}$$

$$= \left\{ y\begin{bmatrix} -1 \\ 1 \end{bmatrix} \;\middle|\; y \text{ real } \# \right\}$$

It is called Subspace of the eigenvectors for the eigenvalue 3, OR eigenspace for $\lambda = 3$.

A basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda = 3$.

(b) $\lambda = 7$

We work effectively: $\begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

y parameter

$x - \frac{1}{3}y = 0$. $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y \\ y \end{bmatrix} = y\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Basis for eigenspace for $\lambda = 7$.

(not from the book.) Ex

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} \qquad \lambda = 3, 1. \text{ (eigenvalues)}$$

Find basis for eigenspaces:

$\lambda = 3$

$$A - 3I = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} & \uparrow & \uparrow \\ & x_2 & x_3 \end{array} \end{array}$$

parameters.

$x_1 - x_2 + x_3 = 0.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Basis for eigenspace for $\lambda = 3$.

The eigenspace for $\lambda = 3$ is 2 dimensional.

$\lambda = 1$

$$A - 1 \cdot I = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 3 \\ 2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + \tfrac{1}{2} x_3 &= 0 \\ x_2 - \tfrac{3}{2} x_3 &= 0 \end{aligned}$$

$\uparrow x_3$ parameter

basis for eigenspace for $\lambda = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} x_3 \\ \frac{3}{2} x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 3/2 \\ 1 \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = \underbrace{2, 2, 2}$

2 is the only eigenvalue

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 free

$x_1 = $ free
$x_2 = 0$
$x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

basis for eigenspace 2.

RULES
for Triangular & Diagonal

upper triangular

$$\begin{bmatrix} \lambda_1 & & * \\ 0 & \lambda_2 & \\ 0 & 0 & \ddots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \longrightarrow \lambda_1, \lambda_2 \ldots \lambda_n \text{ are eigenvalues.}$$

Lower triangular

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ & \lambda_2 & 0 & 0 & 0 \\ & & \ddots & & 0 \\ * & & & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$ diagonal

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \lambda = 1, 2, 5 \text{ eigenvalues.}$$

Thm ⓐ If $\lambda_1, \lambda_2, \ldots \lambda_r$ are distinct eigenvalues of an $n \times n$ matrix A; and $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ are eigenvectors of A where

$\lambda_i \longleftrightarrow v_i (\neq 0)$ $i = 1, 2, \ldots r$

Then

$\{v_1, v_2, \ldots v_r\}$ is a linearly indep set.

ⓑ (It doesn't happen always) but:

if $r = n$, (A $n \times n$ matrix) and everything in ⓐ holds, then $\{v_1, v_2, \ldots v_n\}$ is a basis of $\mathbb{R}^n$.

(Later we will call such a matrix diagonalizable.)

in 5.3 Thm 6 p284.