Review 3.2.

Basic Properties of dets:

Let A and B be $n \times n$ matrices:

Then
- $\det A = \det A^T$
- $\det(A \cdot B) = \det A \cdot \det B$
- A^{-1} exists $\iff \det A \neq 0$
- A^{-1} exists $\implies \det(A^{-1}) = \frac{1}{\det A}$.

(We discussed: $1 = \det I_n = \det(A \cdot A^{-1})$
$= (\det A) \cdot (\det A^{-1})$.

Exc. 22 p.175

Is $\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$ invertible? No, since:

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} \underset{R_1 - 5R_2}{=} \begin{vmatrix} 0 & 15 & 9 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = -\begin{vmatrix} 15 & 9 \\ 5 & 3 \end{vmatrix} \underset{R_1 - 3R_2}{=} -\begin{vmatrix} 0 & 0 \\ 5 & 3 \end{vmatrix} = 0$$

Prop $\det A \neq 0 \overset{(3.2)}{\iff} A^{-1}$ exists

in 2.3 $\begin{cases} \iff \text{The columns of } A \text{ span } \mathbb{R}^n \\ \iff \text{The columns of } A \text{ are lin. indep.} \end{cases}$

in 2.9 $\iff$ The columns of A form a basis of $\mathbb{R}^n$

Exc #24 p175

Is $\left\{ \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} \right\}$ linearly independent?

3 vectors in $\mathbb{R}^3$, so determinants can help.

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 7 & 2 & 6 \end{vmatrix} = -6 \begin{vmatrix} -7 & -3 \\ 2 & 6 \end{vmatrix} - (-5) \begin{vmatrix} 4 & -7 \\ 7 & 2 \end{vmatrix}$$

$$= \underset{\text{neg}}{(-6)}\underset{\text{neg}}{(\underbrace{-42+6})} + \underset{\text{pos.}}{5}\underset{\text{pos}}{(8+49)} > 0$$

$$\neq 0$$

3.2 HW p.175. # 1, 5, 7, 11, 15, 21, 23, 25, 37, 39.

PTO

3.3

Cramer's Rule

How fast can you solve $\begin{cases} 2x + 7y = 11 \\ 3x + 10y = 8 \end{cases}$?

$$x = \frac{\begin{vmatrix} 11 & 7 \\ 8 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix}} = \frac{110-56}{20-21} = -54.$$

$$y = \frac{\begin{vmatrix} 2 & 11 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix}} = \frac{16-33}{20-21} = +17$$

Let A be an $n \times n$ matrix, $\vec{b}$ a column vector.
Define

$$A_i(b) = \left[\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_{i-1} \; b \; \vec{a}_{i+1} \cdots a_n\right],$$

namely $A_i(b)$ is obtained by replacing the i^{th} column of A with b.

Theorem: Cramer's Rule. Let A be an invertible $n \times n$ matrix.

The unique solution of $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i = \frac{\det A_i(b)}{\det A}, \text{ for } i = 1, 2, \dots n.$$

Exc #6 p184

$$2x_1 + x_2 + x_3 = 4$$
$$-x_1 \quad + 2x_3 = 2$$
$$3x_1 + x_2 + 3x_3 = -2$$

① 3 equations, 3 unknowns $\}$ square

$$\begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} \underset{C_3+2C_1}{=} \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 0 \\ 3 & 1 & 9 \end{vmatrix} = +1 \begin{vmatrix} 1 & 5 \\ 1 & 9 \end{vmatrix} = 4 \neq 0$$

② Cramer's Rule applies.

$$\begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} \underset{C_3-C_1}{=} \begin{vmatrix} 4 & 1 & -3 \\ 2 & 0 & 0 \\ -2 & 1 & 5 \end{vmatrix} = -2 \begin{vmatrix} 1 & -3 \\ 1 & 5 \end{vmatrix} = (-2)(8) = -16$$

$$\begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = (12 + 24 + 2) - (6 - 8 - 12) = 38 + 14 = 52$$

$$\begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} \underset{C_3+2C_1}{=} \begin{vmatrix} 2 & 1 & 8 \\ -1 & 0 & 0 \\ 3 & 1 & 4 \end{vmatrix} = +1 \begin{vmatrix} 1 & 8 \\ 1 & 4 \end{vmatrix} = -4.$$

$$x_1 = \frac{-16}{4} = -4$$

$$x_2 = \frac{52}{4} = 13$$

$$x_3 = \frac{-4}{4} = -1$$

p184 Exc #10

$$\left.\begin{aligned} 2s x_1 + x_2 &= 1 \\ 3s x_1 + 6s x_2 &= 2 \end{aligned}\right\}$$

- Find s for which SLE has a unique solution
- Describe soln.

want

$$0 \neq \begin{vmatrix} 2s & 1 \\ 3s & 6s \end{vmatrix} = 12s^2 - 3s = 3s(4s-1)$$

Need $s \neq 0$ and $s \neq \frac{1}{4}$.

Soln

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 6s \end{vmatrix}}{3s(4s-1)} = \frac{6s-2}{3s(4s-1)}$$

$$y = \frac{\begin{vmatrix} 2s & 1 \\ 3s & 2 \end{vmatrix}}{3s(4s-1)} = \frac{4s-3s}{3s(4s-1)} = \frac{s}{3s(4s-1)}$$

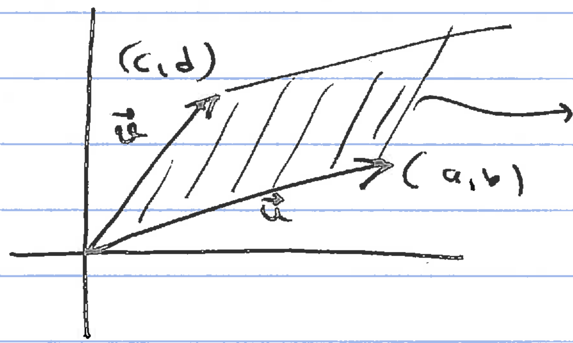
We require $S \neq 0$; $y = \frac{1}{3(4s-1)}$

Caution if $s=0$, the system has NO SOLUTION

$$s=0 \implies \begin{cases} x_2 = 1 \\ 0 = 2 \end{cases}$$ which is inconsistent.

Det / Area

$n=2$



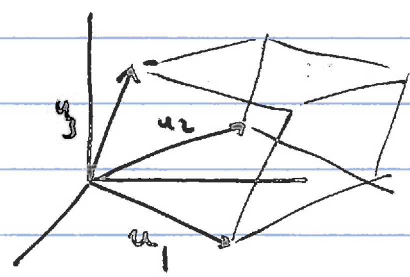
$$\text{area} = \left| \begin{vmatrix} a & c \\ b & d \end{vmatrix} \right| = |ad-bc|$$

D

Exc #20 p 184 vertices $(0,0), \underbrace{(-1,5)}_{u}, \underbrace{(4,-5)}_{v}, \underbrace{(3,-2)}_{u+v}$

$$\text{Area} = \left| \begin{vmatrix} -1 & 4 \\ 3 & -5 \end{vmatrix} \right| = |5-12| = 7.$$

$n=3$



$$\text{volume} = \left| \, |\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3| \, \right|$$

Exc #24. (Caution about adjacent vertices)

$$\begin{vmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{vmatrix} \underset{C_2+2C_3}{=} \begin{vmatrix} 1 & -4 & -1 \\ 4 & -1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix}$$

$$= -(-1+16)$$

$$= -15$$

volume $= 15$.

3.3 HW p184 #1, 3, 5, 7, 14, 23