1. For each of the digraphs below,

(i) Determine if the digraph is strongly connected. State your conclusion. If it is not, explain why. If it is, provide a complete closed path.

(ii) Determine if the digraph is unilaterally connected. State your conclusion. If it is not, explain why. If it is, provide a complete path.

(i) This is a strongly connected digraph since it admits a complete closed path: 
  \[ \text{b e f c b e d a b} \]

(ii) Every strongly connected digraph is unilaterally connected. Since the question is asking you to provide a complete path, you need to state one; for example, you can use the same answer from (i), provided that you state it as: \[ \text{b e f c b e d a b} \] (1pt)

(ii) Since there is no outgoing arc from h (and there is no incoming arc to g), there cannot exist a path from h to g. So, this digraph is not strongly connected.

(ii) This digraph is unilaterally connected since it has a complete path: 
  \[ g j k l i h \]
2. For the digraph $D$ below:
   i. Find the subgraph generated by the vertices $a, b, c$ and $d$.
   ii. Find a strongly connected subgraph of $D$ which is not a strong component of $D$.
   iii. Find all strong components of $D$, and draw them as digraphs. (A list of vertices is NOT sufficient.)
   iv. Draw the condensation digraph $D^*$.
   v. Find a vertex basis for $D$.
   vi. Determine the number of distinct vertex bases of $D$.

- $D^*$ is strongly connected but not a component.
- $D^*$ vertices basis for $D^*$ \{II, III\} was not the question.
- $D^*$ vertex basis for $D$ : \{b, g\}
- $D$ vertex basis for $D$ is obtained by choosing one of \{b, c, f\} and \{g, h, s\}.

- There are 6 possible choices.
3.

The diagram shows a directed graph with three vertices labeled 1, 2, and 3.

a. Find the adjacency, reachability and distance matrices of the digraph above.

\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad D_{\text{Dist}} = \begin{pmatrix} 0 & 1 & 1 \\ 8 & 0 & 1 \\ 8 & 1 & 0 \end{pmatrix} \]

b. Find the number of paths of nonnegative length of at most 2 between all pairs of vertices. Write them in the table below.

\[
\begin{array}{c|ccc}
\text{To} & 1 & 2 & 3 \\
\hline
\text{From} & \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\end{array}
\]

The table shows the number of paths of length at most 2 between vertices 1, 2, and 3.
4.

a. Does the following graph have an Eulerian chain? Explain the reasons for your answer. If there are Eulerian chains (closed or not) in this graph, then find one by showing it on the graph by numbering the edges in the order used and clearly indicate the initial and terminal points of the chain.

\[ \text{Degree}(1) = 3 \]
\[ \text{Degree}(4) = 5 \]
all of the other degrees are even.

By Euler's Thm, this graph admits an Eulerian chain, starting at 6 (or 1).

b. Find a spanning tree for the following graph by using Breadth-First Algorithm, starting at vertex 1. Number the vertices in the order they are chosen, and show your tree on the graph by indicating the edges used.
5. For each of the following statements, (i) first state the statement is correct or false, and then (ii) either provide a proof of the statement if it is correct, or provide a counterexample without a proof if it is false. Assume that there are no loops and no multiple arcs/edges.

You may use any result we gave in class or stated in the textbook, except the one you are actually proving. In other words, saying “this is a theorem in the book” or “this follows the proof we did in the class” receive a maximum of zero points. Furthermore, a slightly different wording or a part of a statement does not constitute a different proposition or theorem.

a. For all vertices $u, v$ and $w$ in a digraph $D$, if $v$ is reachable from $u$ and $w$ is reachable from $v$, then $d(u, w) \leq d(u, v) + d(v, w)$.

\textbf{Correct}

See page 33, Theorem 2.2

\bigskip

b. If a digraph $D$ has a vertex $u_0$ that can reach all of the other vertices of $D$, then $D$ is unilaterally connected.

\textbf{False}

\begin{center}
\begin{tikzpicture}
\node at (-1,0) (u0) {$u_0$};
\node at (0,-1) (u1) {$u_1$};
\node at (1,0) (u2) {$u_2$};
\draw (u0) -- (u1);
\draw (u0) -- (u2);
\end{tikzpicture}
\end{center}

\textit{(Converse is true, page 37, Lemma.)}

\bigskip
c. Let $A$ be the adjacency matrix of a digraph $D$. Then $(A^k)_{ij}$ is the number of paths of length $k$ from $u_i$ to $u_j$ in $D$. ($M_{ij}$ denotes the $i$th row $j$th column entry of a matrix $M$).

\textbf{Correct}

See page 54, Theorem 2.11