1. Let x and y be arbitrary real numbers. For each of the following statements,

(i) State whether it is true or false, and

(ii) Prove the statement if it is true, or give a counterexample disproving the statement with details if it is false.

If (i) is not stated, then (ii) will not earn credit.

a. Let x and y be arbitrary real numbers. If $\underbrace{x + y \text{ is irrational}}_{p}$, then $\underbrace{x \text{ is irrational or } y \text{ is irrational}}_{q}$.

(i) TRUE

(ii) Prove the contrapositive: $\sim q \Rightarrow \sim p$.

x is not irrational and y is not irrational $\Rightarrow$ x+y is not irrational

Suffices to prove if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ then $x+y \in \mathbb{Q}$.

Assume $x, y \in \mathbb{Q}$. $\exists p, q, r, s \in \mathbb{Z}$, $q \neq 0, s \neq 0$ s.t.

$x = \frac{p}{q}$, and $y = \frac{r}{s}$. Then $x+y = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}$

$q \neq 0, s \neq 0 \Rightarrow sq \neq 0$.

$p, q, r, s \in \mathbb{Z} \Rightarrow ps+qr \in \mathbb{Z}$ and $sq \in \mathbb{Z}$. So $\frac{ps+qr}{qs} \in \mathbb{Q}$

hence $x+y \in \mathbb{Q}$.

b. Let x and y be arbitrary real numbers. If x is irrational and y is rational, then xy is irrational.

(i) False

It suffices to find x irrational, and y rational but xy is rational.

Ex. $x = \sqrt{2}$ proved irrational in class.

$y = 0$

$xy = 0$.

2. Let A, B, C, D and E be subsets of a universal set U.

a. Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

You are asked to prove Theorem 2.1.13(g).You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.1.13(g) or its consequences. Simply, saying "This follows a theorem in the book or lecture" will not earn any credit. You are expected to provide a proof.

$\forall x \in U$

$x \in A \setminus (B \cup C) \iff x \in A$ and $x \notin (B \cap C)$

$\iff x \in A$ and not $(x \in B \cap C)$

$\iff x \in A$ and not$(x \in B$ and $x \in C)$

$\iff x \in A$ and $(x \notin B$ OR $x \notin C)$

$\iff (x \in A$ and $x \notin B)$ OR $(x \in A$ and $x \notin C)$

$\iff x \in A \setminus B$ OR $x \in A \setminus C$

$\iff x \in (A \setminus B) \cup (A \setminus C)$.

b. Prove that if $E \subseteq D \subseteq U$, then $(U \setminus D) \subseteq (U \setminus E)$.

$E \subseteq D \iff \forall x \in U (x \in E \Rightarrow x \in D)$

$\iff \forall x \in U (x \notin D \Rightarrow x \notin E)$ Contrapositive

$\forall x \in U \setminus D$, one has $x \in U$ and $x \notin D$

hence $x \in U$ and $x \notin E$

$x \in U \setminus E$.

We proved that

$\forall x \quad x \in U \setminus D \Rightarrow x \in U \setminus E$.

Hence $U \setminus D \subseteq U \setminus E$.

3. DO ANY TWO OF (a), (b) and (c).

a. Give the definition of a function $f : A \to B$. You may assume that the relation $f \subseteq A \times B$ and f is non-empty.

A non-empty relation $f \subseteq A \times B$ is called a function if

existence (i) $\forall a \in A \ \exists b \in B$ s.t. $(a,b) \in f$ (i.e. $f(a) = b$)

uniqueness (ii) $\forall a \in A \ \forall b, c \in B$ $(a,b) \in f$ and $(a,c) \in f \Rightarrow b = c$.

b. Let $f : A \to B$ be a function. Prove that for every subset $D \subseteq B$, one has $f[f^{-1}(D)] \subseteq D$.

You are asked to prove Theorem 2.3.16(b). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.3.16(b) or its consequences. Simply, saying "This follows a theorem in the book or lecture" will not earn any credit. You are expected to provide a proof.

To prove $f(f^{-1}(D)) \subseteq D$, it suffices to prove $\forall b \in f(f^{-1}(D))$, one has $b \in D$.

(If $f(f^{-1}(D)) = \emptyset$, then there is nothing to prove.

Let $b \in f(f^{-1}(D))$ be arbitrary.

(Recall defn: $y \in f(C)$ iff $y = f(x)$ for some $x \in C$)

So $\exists a \in f^{-1}(D)$ s.t. $b = f(a)$

(Recall defn $x \in f^{-1}(D)$ iff $f(x) \in D$)

$a \in f^{-1}(D) \Rightarrow f(a) \in D$. But $b = f(a) \in D$. So $b \in D$.

c. Let $f : A \to B$ be a function. Prove that if $f[f^{-1}(D)] = D$ for every subset $D \subseteq B$, then f is surjective.

method Many ways to do this proof

I $\forall D \subseteq B$, $f(f^{-1}(D)) = D$.

Take $D = B$

$f(f^{-1}(B)) = B$.

$f^{-1}(B) = \{x \in A \mid f(x) \in B\} = A$ defn of function.

$f(A) = B$. means surjective.

method II Suppose f is not surjective: $\exists b \in B$, $b \notin$ range f. Take $D = \{b\}$

$f^{-1}(\{b\}) = \emptyset$, $f(\underbrace{f^{-1}(\{b\})}_{\emptyset}) = \emptyset$.

Contradiction since $f(f^{-1}(\{b\})) = \{b\}$.

III Take any $y \in B$, Let $D = \{y\}$

$f(f^{-1}(\{y\})) = \{y\} \neq \emptyset$. So $f^{-1}(\{y\}) \neq \emptyset$.

So $\exists x \in f^{-1}(\{y\})$ s.t. $f(x) = y$.

4. a. State *The Principle of Mathematical Induction.*

Let $p(n)$ be a statement for each $n \in \mathbb{N}$.

If (i) $P(1)$ is true, and (ii) $\forall k \in \mathbb{N}\ (p(k) \Rightarrow p(k+1))$, then

$p(n)$ is true for all $n \in \mathbb{N}$.

b. Prove *The Principle of Mathematical Induction,* by using *The Well-Ordering Property of* $\mathbb{N}$.

You are asked to prove Theorem 3.1.2. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 3.1.2 or its consequences. Simply, saying "This follows a theorem in the book or lecture" will not earn any credit. You are expected to provide a proof.

Proof by Contradiction.

Suppose ($p(n)$ is true for all $n \in \mathbb{N}$) is false.

$\exists n_0 \in \mathbb{N}$ s.t. $p(n_0)$ is false.

Let $S = \{ n \in \mathbb{N} \mid p(n) \text{ is false.} \}$

$S \neq \emptyset$, since $n_0 \in \mathbb{N}$ and $p(n_0)$ is false: $n_0 \in S$.

$S \subseteq \mathbb{N}$, Hence by The well-ordering of $\mathbb{N}$:

$\forall S \subseteq \mathbb{N}\ (S \neq \emptyset \Rightarrow \exists m \in S, \forall n \in S,\ n \geq m)$.

that is $\exists m = \min S$.

By (i) $P(1)$ true, so $m \neq 1$; $m \geq 2$ since $m \in S \subseteq \mathbb{N}$.

$m-1 \in \mathbb{N}$, but $m-1 \notin S$ since $m = \min S$.

$P(m-1)$ is true by defn of S.

by (ii) $P(m-1) \Rightarrow p(m)$

$p(m)$ is true, which contradicts with $m \in S$

& $S = \{ n \in \mathbb{N} \mid p(n) \text{ is false.} \}$.

There are 2 versions:
Same statements, but different order.

Check
- OR
- or

next page

5. TRUE OR FALSE

CIRCLE YOUR ANSWERS. NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.

Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: Read very carefully.

TRUE FALSE

a. The negation of the statement $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R}, (x < 5 \Rightarrow \exists z \in \mathbb{R} \ni (x + z = 1 \text{ and } y > z))$ is equivalent to $\exists x \in \mathbb{R} \ni \exists y \in \mathbb{R} \ni (x \geq 5 \text{ and } (\forall z \in \mathbb{R}, (x + z \neq 1 \text{ or } y \leq z)))$.

no

negation is

$\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x < 5 \text{ and} (\forall z \in \mathbb{R}\ (x+z \neq 1 \text{ or } y \leq z)))$

TRUE FALSE **b.** The set of rational numbers $\mathbb{Q}$ is an uncountable set since it is an infinite set.

$\mathbb{Q}$ is countable and infinite

TRUE FALSE **c.** For every non-empty set A, in order to prove "$\exists n \in A \ni p(n)$" is false, it suffices to show that "$\forall n \in A, \sim p(n)$" is true.

TRUE FALSE **d.** $\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, \forall z \in \mathbb{R},\ z > y$ implies that $z > x + y$.

$\exists x = 0\ \forall y\ \forall z\quad z > y \Rightarrow z > x + y = y$

TRUE FALSE **e.** Every subset of a denumerable set is denumerable.

$\emptyset \subseteq \mathbb{N}$. or $\{1\} \subseteq \mathbb{N}$

$\emptyset$ is finite but not denumerable.

$\{1\}$ is " " " " .

Check
OR
or

There are two versions
Same statements, but different order

5. **TRUE or FALSE**

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HINT: Read very carefully.

TRUE FALSE

a. The negation of the statement $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (x < 5 \Rightarrow \exists z \in \mathbb{R} \ni (x + z = 1 \text{ and } y > z))$ is equivalent to $\exists x \in \mathbb{R} \ni \exists y \in \mathbb{R} \ni (x \geq 5 \text{ and } (\forall z \in \mathbb{R}, (x + z \neq 1 \text{ or } y \leq z)))$.

TRUE FALSE **b.** For every non-empty set A, in order to prove "$\exists n \in A \ni p(n)$" is false, it suffices to show that "$\forall n \in A, \sim p(n)$" is true.

TRUE FALSE **c.** The set of rational numbers $\mathbb{Q}$ is an uncountable set since it is an infinite set.

TRUE FALSE **d.** Every subset of a denumerable set is denumerable.

TRUE FALSE **e.** $\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, \forall z \in \mathbb{R}$, $z > y$ implies that $z > x + y$.