1. a. Complete the following ε-N definition (Definition 4.1.2) for a convergent sequence:
Definition:
A sequence (s_n) is said to converge to the real number L provided that....

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \in \mathbb{N} \ (n \geq N \Rightarrow |s_n - L| < \varepsilon)$$

b. By using **only** the ε-N definition of convergence and inequalities, prove that
$$\lim_{n\to\infty} \frac{4n-2}{2n+3} = 2.$$
(Using limit theorems (4.2) will not earn credit.)

Let $\varepsilon > 0$ be given.
Choose $N \in \mathbb{N}$ s.t. $N > \frac{4}{\varepsilon}$.
$\forall n \in \mathbb{N}$, $n \geq N$, we have

$$\left|\frac{4n-2}{2n+3} - 2\right| = \left|\frac{4n-2-4n-6}{2n+3}\right| = \frac{8}{2n+3} \underset{\substack{\uparrow \\ \text{since } 2n+3 \geq 2n}}{\leq} \frac{8}{2n} = \frac{4}{n} \underset{\substack{\uparrow \\ \text{since } n \geq N}}{\leq} \frac{4}{N} \underset{\substack{\uparrow \\ \text{since } N > \frac{4}{\varepsilon}}}{<} \varepsilon$$

2. a. State The Density of Rational Numbers in Real Numbers.

$\forall x, y \in \mathbb{R}$, if $x < y$, then $\exists r \in \mathbb{Q}$ s.t. $x < r < y$.

b. Prove that if x and y are real numbers with $x < y$, then there exists an irrational number a such that $x < a < y$.

You are asked to prove Theorem 3.3.15. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 3.3.15 or its consequences. Simply, saying "This follows Theorem 3.3.15" will not earn any credit. You cannot use the result of any exercise unless you provide a solution of that exercise.

Let $x < y$, $x, y \in \mathbb{R}$ be given.

$\sqrt{2} \notin \mathbb{Q}$, as proved in class; $\sqrt{2} > 0$.

$\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$. By (a) $\exists r = \frac{p}{q} \in \mathbb{Q}$ (where $p, q \in \mathbb{Z}$, $q \neq 0$)

s.t. $\frac{x}{\sqrt{2}} < \frac{p}{q} < \frac{y}{\sqrt{2}}$.

if $p/q \neq 0$, then

$x < \frac{p\sqrt{2}}{q} < y$.

$\frac{p\sqrt{2}}{q} \notin \mathbb{Q}$, [otherwise; $\frac{p\sqrt{2}}{q} \in \mathbb{Q}$

$\frac{p\sqrt{2}}{q} = \frac{r}{s} \Rightarrow \sqrt{2} = \frac{qr}{ps} \in$ contradiction.)

If $p/q = 0$, then $\exists \frac{p'}{q'} \in \mathbb{Q}$

s.t. $\frac{x}{\sqrt{2}} < 0 < \frac{p'}{q'} < \frac{y}{\sqrt{2}}$.

$\frac{p'\sqrt{2}}{q'} \notin \mathbb{Q}$.

c. Prove that if x and y are real numbers with $x < y$, then there exists infinitely many irrational numbers between x and y.

Given $x, y \in \mathbb{R}$, $x < y$.

Let $P(n)$ be the statement $\exists a_1, a_2, \dots a_n$ s.t.

$x < a_1 < a_2 < \dots < a_n < y$, $\forall i = 1, \dots, n$, $a_i \in \mathbb{R} - \mathbb{Q}$.

Proof by induction. $P(1)$ was proved in (b) above.

To prove $P(k) \Rightarrow P(k+1)$. Assume induction hypothesis:

$P(k)$ $\exists a_1 \dots a_k \in \mathbb{R} - \mathbb{Q}$ s.t. $x < a_1 < a_2 < \dots < a_k < y$.

By (b) or $P(1)$, $\exists a_{k+1} \in \mathbb{R} - \mathbb{Q}$ s.t. $a_k < a_{k+1} < y$.

Hence $\exists a_1, \dots a_{k+1} \in \mathbb{R} - \mathbb{Q}$ s.t. $x < a_1 < a_2 < \dots < a_k < a_{k+1} < y$.

$P(k+1)$ is established. Induction step is proved.

$\forall n$ $P(n)$ is true, and hence there are at least n distinct irrational numbers between x & y.

Since this is true $\forall n \in \mathbb{N}$, the cardinality of irrational numbers between x & y cannot be finite,

Hence $\exists$ infinitely many irrationals between x & y.

3. Fill in your answers or circle your answers. You are not expected to provide justifications. No partial credit. No answers, incomplete or wrong answers receive 0 points, no penalty in this question.

Leaving an answer blank will not earn credit. If an answer is the empty set, then state it so. If there exists no answer, then state it so, such as: *DNE (does not exist)*.

Consider all of the sets in this question as subsets of $\mathbb{R}$.

Caution: For each part make sure to use the correct set.

Let $S = (0,3) \cup (3,4) \cup \{6\}$.

a. The maximum of S : $\max S =$ 6

b. The infimum of S : $\inf S =$ 0

c. The interior of S : *int* $S =$ $(0,3) \cup (3,4)$

d. The minimum of S : $\min S =$ DNE

e. The closure of S : *cl* $S =$ $[0,4] \cup \{6\}$

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Let $T = \{0\} \cup (\mathbb{Q} \cap [1,2))$

f. The supremum of T : $\sup T =$ 2

g. The accumulation points of T : $T' =$ $[1,2]$

h. The boundary of T : *bd* $T =$ $[1,2] \cup \{0\}$

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Circle your answer. Let $U = \bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$. Hint: Simplify U first.

$U = \{0\}$ by Archimedean principle.

i. Is U an open set? Yes or No

j. Is U a closed set? Yes or No

k. Is U a bounded set? Yes or No

$\forall S \subseteq \mathbb{R} \, (S \neq \emptyset$ and $\exists M \, \forall s \in S, \, s \leq M \implies \exists \sup S \in \mathbb{R})$

4. a. State the following.
The Completeness Axiom for ℝ (The Least Upper Bound property of ℝ):

For every non-empty subset S of $\mathbb{R}$, where S is bounded above, there exists a supremum of S in $\mathbb{R}$.

b. Prove that every bounded and monotone increasing sequence (s_n) is convergent, by using The Completeness Axiom for ℝ.

You are asked to prove a version of Theorem 4.3.3. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 4.3.3 or its consequences. Simply, saying "This follows Theorem 4.3.3" will not earn any credit. You cannot use the result of any exercise unless you provide a solution of that exercise.

Let (s_n) be a sequence in $\mathbb{R}$ s.t.

(i) $\forall n \;\; s_n \leq s_{n+1}$ and

(ii) $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, \;\; |s_n| \leq M$.

We want to prove: there is $L \in \mathbb{R}$ s.t. $\lim_{n\to\infty} s_n = L$.

Let $S = \{s_n \mid n \in \mathbb{N}\}$ be the set of values of (s_n).

$S \neq \emptyset$, since (s_n) is given.

S is bounded above by M: $\quad s_n \leq |s_n| \leq M, \; \forall n \in \mathbb{N}$

$\exists \sup S = L \in \mathbb{R}$ by Completeness Axiom.

Let $\varepsilon > 0$ be given. $L - \varepsilon$ is not an upper bound for S, since $L - \varepsilon < L = \sup S$.

$\exists x \in S$ s.t. $\quad L - \varepsilon < x \leq L$.

$x = s_N$ for some N, since S is the set of the values of (s_n).

$\forall n \in \mathbb{N}, \; n \geq N, \quad s_n \geq s_N$ since (s_n) is increasing.

Hence $\forall n \geq N$

$$L - \varepsilon < x = s_N \leq s_n \leq L < L + \varepsilon$$

$\uparrow L = \sup S$

$$L - \varepsilon < s_n < L + \varepsilon \implies -\varepsilon < s_n - L < \varepsilon$$
$$\implies |s_n - L| < \varepsilon.$$

$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N} \, (n \geq N \implies |s_n - L| < \varepsilon)$. $\lim_{n\to\infty} s_n = L$.

There are 2 versions Check "or" "OR"
Same questions, different order.

5. TRUE or FALSE.

CIRCLE YOUR ANSWERS. SHOW NO WORK.

PTO

Correct answers are +4 points each, wrong answers are -1 points each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5 will be included your total grade only if it is positive.

TRUE FALSE a. Let $S \subseteq \mathbb{R}$. If $L = \sup S$ for some $L \in \mathbb{R}$, then $L > x$ for all x in S.

need $L \geq x$

for example: $S = \{1\}$ $\sup S = 1$.

TRUE FALSE b. Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $|s_n| \leq |t_n|, \forall n \in \mathbb{N}$.
If (s_n) diverges to $-\infty$, then (t_n) diverges.

Suppose (t_n) converges. Then t_n is bounded: $\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, 0 \leq |t_n| \leq M$. So, $\forall n$ $|s_n| \leq |t_n| \leq M$. Since $s_n \to -\infty$ $\exists N$ $\forall n \geq N$ $s_n < -(M+1)$, which implies $|s_n| > M+1$ X

TRUE FALSE c. Let (s_n) be a sequence of rational numbers, $\mathbb{Q}$.
If $\forall \varepsilon > 0, \exists M \ni \forall m,n \in \mathbb{N} \ni (m,n \geq M \Rightarrow |s_n - s_m| < \varepsilon)$, Cauchy
* then $\exists L \in \mathbb{R}$, $\ni \forall \varepsilon > 0, \exists M \ni \forall n \in \mathbb{N}, (n \geq M \Rightarrow |s_n - L| < \varepsilon)$. Convergent

(s_n) Cauchy in $\mathbb{Q}$, and also Cauchy in $\mathbb{R}$.
(s_n) Cauchy in $\mathbb{R}$ $\Longrightarrow$ (s_n) convergent in $\mathbb{R}$.
Every sequence in $\mathbb{Q}$ is also a sequence in $\mathbb{R}$.

TRUE FALSE d. Let (s_n) and (t_n) be convergent sequences of positive real numbers.
Then $\left(\frac{s_n}{t_n}\right)$ converges.

$$\left.\begin{array}{l} s_n = \frac{1}{n} \to 0 \\ t_n = \frac{1}{n^2} \to 0 \end{array}\right\} \quad \frac{s_n}{t_n} = n \text{ doesn't converge.}$$

(We can't use Thm 4.2.1, since $\lim t_n = 0$.

TRUE FALSE e. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}$ (if $x \neq 0$ then $\exists n \in \mathbb{Z}$ (integers) such that $nx > y$).

If $x > 0$, then this follows Archimedean principle

If $x < 0$ then $-x > 0$, $\exists m \in \mathbb{N}$ s.t. $m(-x) > y$] A.P.

$(-m)x > y$, where $\underbrace{-m}_{n} \in \mathbb{Z}$.

There are 2 versions.
Same questions, different order.

5. TRUE OR FALSE.

CIRCLE YOUR ANSWERS. SHOW NO WORK.

Correct answers are +4 points each, wrong answers are -1 points each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5 will be included in your total grade only if it is positive.

TRUE FALSE **a.** $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}$ (if $x \neq 0$ then $\exists n \in \mathbb{Z}$ (integers) such that $nx > y$).

TRUE FALSE **b.** Let $S \subseteq \mathbb{R}$. If $L = \sup S$ for some $L \in \mathbb{R}$, then $L > x$ for all x in S.

TRUE FALSE **c.** Let (s_n) and (t_n) be convergent sequences of positive real numbers. Then $\left(\frac{s_n}{t_n}\right)$ converges.

TRUE FALSE **d.** Let (s_n) be a sequence of rational numbers, $\mathbb{Q}$.
If $\forall \varepsilon > 0, \exists M \ni \forall m,n \in \mathbb{N} \ni (m,n \geq M \Rightarrow |s_n - s_m| < \varepsilon)$,
then $\exists L \in \mathbb{R}, \ni \forall \varepsilon > 0, \exists M \ni \forall n \in \mathbb{N}, (n \geq M \Rightarrow |s_n - L| < \varepsilon)$.

TRUE FALSE **e.** Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $|s_n| \leq |t_n|, \forall n \in \mathbb{N}$.
If (s_n) diverges to $-\infty$, then (t_n) diverges.