1. a. Find the strong components of the stochastic digraph $D$ above, and draw its condensation digraph.

b. Identify all transient sets.

c. Identify all ergodic sets.

d. Identify all closed sets.
2. You do not need to give any explanations in this question, but there is no partial credit either.

For each of the following digraphs of Markov chains, do all of the following.

i. Determine whether the Markov chain ergodic or not.

ii. Determine whether the Markov chain regular or not.

iii. Find $k$ such that $P^k$ has all positive entries, where $P$ is the probability matrix of the Markov chain. If this is not possible, state it so.

iv. Determine whether the Markov chain is absorbing or not.

a. 

b. 

c.
3. a. Find the complement \( G^c \) of the graph \( G \) below, and put a transitive orientation on \( G^c \).

b. Is the graph \( G \) an interval graph? Find an interval assignment \( F \) for \( G \), if it exists. Explain why not, if does not exist.
Let an autonomous pulse process be defined by the weighted digraph above with the starting values \( V(\text{start}) = (0, 0, 0, 0) \).

a. Determine if this process is pulse stable under all simple pulses. If not all, then list the simple pulses under which this pulse process is pulse stable. If none, state it so.

b. Determine if this process is value stable under all simple pulses. If not all, then list the simple pulses under which this pulse process is value stable. If none, state it so.
5. Suppose that each man has only one son, and the heights are listed as short, medium or tall. Suppose that a tall father will have a tall son with probability 0.8, a medium-height son with probability 0.2, and a short son with probability 0. A medium-height father will have a tall, medium-height, or short son with probabilities 0.1, 0.7, and 0.2 respectively. A short father will have a tall, medium-height, or short son with probabilities 0, 0.4, and 0.6 respectively.

a. Describe the transition matrix and digraph for this Markov chain.

b. What is the probability that the grandson of a tall man will be short?

c. Suppose that we know that 20% of the male population is tall, 20% short and the rest is medium now. What will be the distribution of heights 2 generations from now?

d. If we assume that this trend continues, what will be the distribution of heights in the long run?
6. Let \( A = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 3 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \end{bmatrix} \)

a. Draw the Coates digraph \( D \) of the matrix \( A \) and find the strong components of the digraph \( D \).

b. Find all eigenvalues of \( A \) (with multiplicities and including the complex roots) and show their relation to the strong components of \( D \).
Let an autonomous pulse process be defined by the weighted digraph above, with the starting values \( V(-1) = V(\text{start}) = (0, 0, 0) \).

a. Calculate the pulses \( P(k) \) and values \( V(k) \) for \( k = 0, 1, \) and \( 2 \), for an initial pulse \( P(0) = (0, 0, 1) \).

b. Under the simple pulse starting at \( u_3 \), find \( p_3(2) \) and \( v_3(2) \) if \( v_3(\text{start}) = 0 \).

c. Suppose that both vertices \( u_1 \) and \( u_3 \) go up 1 unit at time \( k = 0 \), what happens to the vertex \( u_2 \) at time \( k = 2 \)?
7. (25 points) For each of the following 4 statements:
   (i) State whether it is true or false, and
   (ii) either prove the statement if it is true, or
   provide a counterexample without a proof if it is false.
   Assume that there are no multiple arcs/edges. You may use any result we gave in class or stated
   in the textbook, except the one you are actually proving. In other words, saying "this is a theorem in
   the book" or "this follows the proof we did in the class" as the whole proof will receive a maximum
   of zero points. Furthermore, a slightly different wording or a part of a statement does not constitute a
different proposition or theorem.

   a. A Markov chain with a strongly connected transition digraph is regular.

   b. All eigenvalues of a digraph with no cycles (hence no loops) are 0.

   c. If a pulse process has an eigenvalue 1, then under every simple pulse it is not value stable.

   d. If $A$ and $B$ are $n \times n$ stochastic matrices, then $AB$ is a stochastic matrix.
7. For each of the following 3 statements:
   (i) State whether it is true or false, and
   (ii) either prove the statement if it is true, or
   provide a counterexample without a proof if it is false.

   Assume that there are no multiple arcs/edges. You may use any result we gave in class or stated in the textbook, except the one you are actually proving. In other words, saying "this is a theorem in the book" or "this follows the proof we did in the class" as the whole proof will receive a maximum of zero points. Furthermore, a slightly different wording or a part of a statement does not constitute a different proposition or theorem.

   a. A Markov chain with a strongly connected transition digraph is ergodic.

   b. Every ergodic Markov chain is regular.

   c. If $A$ and $B$ are $n \times n$ stochastic matrices, then $AB$ is a stochastic matrix.