MATH:2850
MIDTERM 1
March 3, 2017

NAME. SOLUTION

SIGNATURE.______________________

Do all **5** problems, **20 points each.**

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example: $\sqrt{2}$, $\ln 2$, e^3 or $\sin\frac{\pi}{8}$), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

Do Not Write Below:

1.______

2.______

3.______

4.______

5.______

TOTAL.______

Problem 1.

Let $f(x,y) = -xy$.

For all of the graphs below: label the axes, and label the graphs with their functions.

a. Determine the level curves of the function $f(x,y)$ for the height values of $c = -1, 0, 1$. Sketch these level curves in the same coordinate system, and make sure to indicate the height c of each curve.

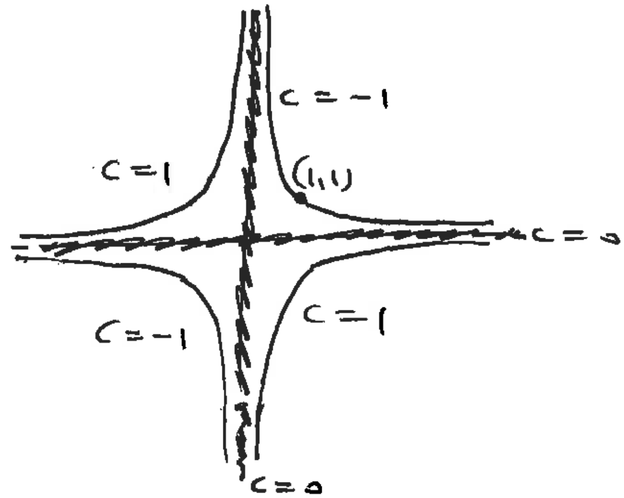
$-xy = -1 \Rightarrow y = \frac{1}{x}$

$-xy = 0 \Rightarrow xy = 0$

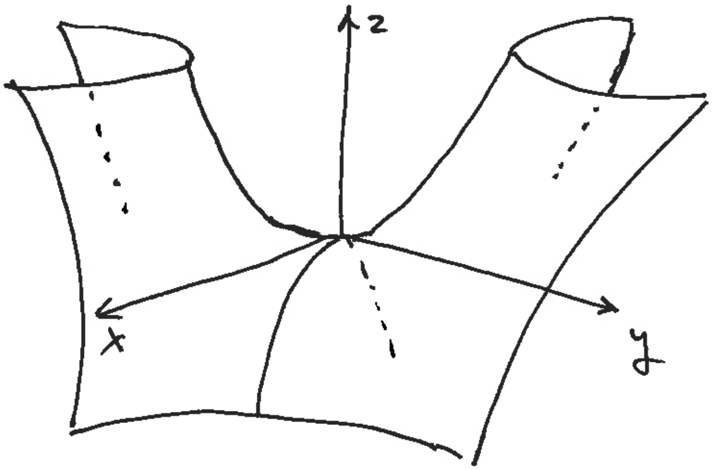
$\Rightarrow x = 0$ or $y = 0$

$-xy = 1 \Rightarrow y = -\frac{1}{x}$

$c = 0$ is the union of both x & y axes.



b. Sketch the (explicit) graph of $z = f(x,y)$. Describe it in words if you can't draw it.



c. Find an equation describing the tangent plane to the (explicit) graph of $z = f(x,y)$ when $x = 3$ and $y = 4$.

$\frac{\partial f}{\partial x} = -y$

$\frac{\partial f}{\partial y} = -x$

$f(3,4) = -12$

$f_x(3,4) = -4$

$f_y(3,4) = -3$

$z = -12 + (-4)(x-3) + (-3)(y-4)$

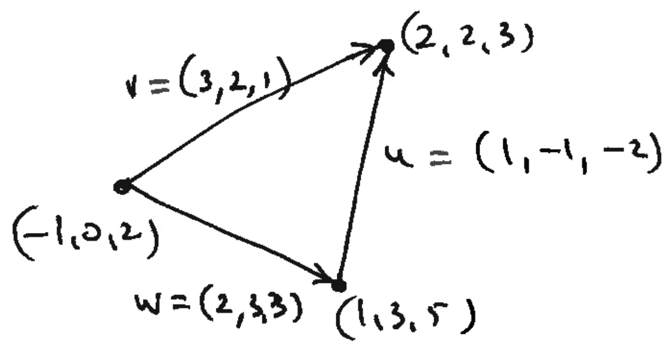
$z = -12 - 4x + 12 - 3y + 12$

$z = -4x - 3y + 12.$

Problem 2.

Let $P = (2,2,3)$, $Q = (-1,0,2)$ and $R = (1,3,5)$ be given three points in $\mathbb{R}^3$.

a. Find a set of parametric equations for the plane that contains the points P, Q, and R.



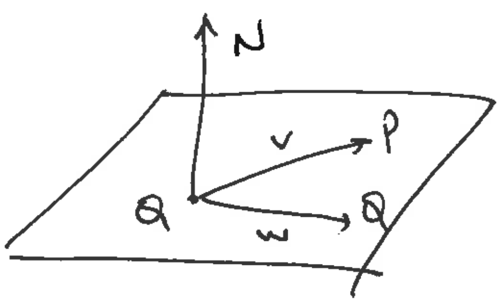
$r(s,t) = (2,2,3) + s(2,3,3) + t(3,2,1$

or

$$x = 2 + 2s + 3t$$
$$y = 2 + 3s + 2t$$
$$z = 3 + 3s + t$$

There are many solutions using pairs of vectors from $\{\vec{u}, \vec{v}, \vec{w}\}$ & using different pts among $\{P, Q, R\}$.

b. Find a closed (non-parametric) coordinate equation for the plane passing through P, Q and R.



$$N = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = (3, -7, 5)$$

$$3x - 7y + 5z = D = (3,-7,5)\cdot(-1,0,2)$$
$$= -3 + 10$$

$$\boxed{3x - 7y + 5z = 7}$$

c. Compute the area of the triangle with vertices P, Q and R.

$$\text{Area} = \frac{1}{2}\|v \times w\| = \frac{1}{2}\|(3,-7,5)\| = \frac{\sqrt{9+49+25}}{2}$$

$$= \frac{\sqrt{83}}{2}.$$

Problem 3. Let $f(x,y) = x\cos xy + y\ln x - 3x + 4$

a. Calculate all first and second order partial derivatives of f.

$$f_x = \cos xy - xy\sin xy + \frac{y}{x} - 3$$

$$f_y = -x^2 \sin xy + \ln x$$

$$f_{xx} = -y\sin xy - y\sin xy - x \cdot y^2 \cos xy + \frac{-y}{x^2}$$

$$f_{xy} = -x\sin xy - x\sin xy - x^2 y\cos xy + \frac{1}{x}$$

$$f_{yx} = -2x\sin xy - x^2 y\cos xy + \frac{1}{x}$$

$$f_{yy} = -x^3 \cos xy$$

b. Calculate $\nabla f(2,0)$.

$$\nabla f(2,0) = (\cos 0 - 0 + 0 - 3,\ \ln 2) = (-2, \ln 2)$$

c. Calculate the directional derivative of f at $(x,y) = (2,0)$ in the direction parallel to the vector $4\mathbf{i} + 3\mathbf{j}$.

$4i + 3j$ is not a unit vector. $\|4i + 3j\| = \sqrt{16+9} = 5$

$u = \dfrac{4i+3j}{5}$ is a unit vector.

$$(D_u f)(2,0) = \underbrace{\left(\tfrac{4}{5}, \tfrac{3}{5}\right)}_{u} \cdot \underbrace{(-2, \ln 2)}_{\nabla f(2,0)} = \frac{-8 + 3\ln 2}{5}$$

Problem 4.
Let $\mathbf{f}(x,y,z) = (-xy - z + x^2,\ yz - 2)$ and $\mathbf{g}(u,v) = (u^2,\ u - 2v,\ uv)$.
Calculate the following: $D\mathbf{f}, D\mathbf{g}$, and $D(\mathbf{g} \circ \mathbf{f})(3,2,1)$.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$D\mathbf{f} =$

2×3

$$\begin{bmatrix} -y + 2x & -x & -1 \\ 0 & z & y \end{bmatrix}$$

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$D\mathbf{g} =$

3×2

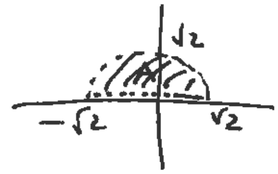
$$\begin{bmatrix} 2u & 0 \\ 1 & -2 \\ v & u \end{bmatrix}$$

$$f(3,2,1) = (-6 - 1 + 9,\ 2 - 2) = (2,0)$$

$D(\mathbf{g} \circ \mathbf{f})(3,2,1) = Dg(f(3,2,1)) \cdot Df(3,2,1)$

$$= Dg(2,0) \cdot Df(3,2,1)$$

$$= \begin{bmatrix} 4 & 0 \\ 1 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -12 & -4 \\ 4 & -5 & -5 \\ 0 & 2 & 4 \end{bmatrix}$$



Problem 5.

a. Let $A = \{(x,y) \in \mathbb{R}^2 \mid y > 0 \text{ and } x^2 + y^2 < 2\}$ be a subset of $\mathbb{R}^2$.
What is the boundary of A? Write it as a set of the form $\{(x,y) \in \mathbb{R}^2 \mid \ldots.\}$

$$\text{Bd } A = \{(x,y) \mid (-\sqrt{2} \le x \le \sqrt{2} \text{ and } y = 0) \text{ OR } (x^2+y^2 = 2 \text{ and } y \geq 0)\}$$

Is A an open set? Circle the correct answer: YES or NO. $(\text{Bd}A) \cap A = \emptyset$

Is A a closed set? Circle the correct answer: YES or NO $\quad \text{Bd} \not\subseteq A$

b. Calculate the following limits. If any of them does not exist, state it so. **Justify your answers.**

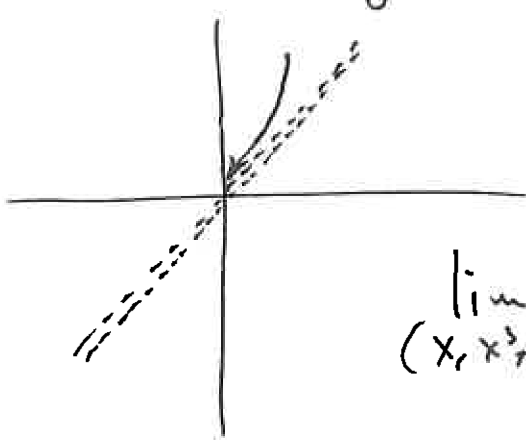
i. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$ since $0 \le y^2 \le x^2 + y^2$

$$0 \le \frac{y^2}{x^2+y^2} \le 1$$

$$0 \le \left| \frac{xy^2}{x^2+y^2} \right| \le |x| \to 0 \text{ as } (x,y) \to (0,0)$$

ii. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x-y} = \underline{\underline{\text{DNE}}}$, since

Domain of $\frac{x^2}{x-y} = \{(x,y) \mid x \neq y\}$.



Approach $(0,0)$ along the curve $y = x^3 + x$ as $x \to 0$

$$\lim_{(x,x^3+x)\to(0,0)} \frac{x^2}{x-y} = \lim_{x\to 0} \frac{x^2}{-x^3}$$

$$= \lim_{x \to 0} -\frac{1}{x} = \text{DNE}.$$

Read the class notes from Feb 9, page 4.