

**MATH 2850**  
**Midterm 2**  
**April 14, 2017**

NAME. SOLUTION

SIGNATURE. \_\_\_\_\_

**Do all 5 problems, 20 points each.**

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example:  $\sqrt{2}$ ,  $\ln 2$ ,  $e^3$  or  $\sin \frac{\pi}{8}$ ), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

**DO NOT WRITE BELOW:**

1. \_\_\_\_\_

2. \_\_\_\_\_

3. \_\_\_\_\_

4. \_\_\_\_\_

5. \_\_\_\_\_

Total. \_\_\_\_\_

$$\nabla f = (e^x(y^2 - x^2 - 2x), 2ye^x)$$

**Problem 1.**

Let  $f(x,y) = e^x(y^2 - x^2)$

a. Calculate  $\nabla f$  and  $H_f$ .

$$H_f = \begin{bmatrix} e^x(y^2 - x^2 - 2x - 2) & 2ye^x \\ 2ye^x & 2e^x \end{bmatrix}$$

b. Calculate the second order Taylor polynomial of  $f$  at  $(x,y) = (0,1)$ .

$$f(0,1) = 1$$

$$\nabla f(0,1) = (1, 2)$$

$$H_f(0,1) = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$P_2 = 1 + 1 \cdot (x-0) + 2(y-1) + \frac{1}{2} \left( -1(x-0)^2 + 4(x-0)(y-1) + 2(y-1)^2 \right)$$

c. Identify all critical points of  $f$  and determine the nature of each critical point (local maxima, local minima, saddle points).

$$\nabla f = 0 \Leftrightarrow \begin{cases} e^x(y^2 - x^2 - 2x) = 0 \\ 2ye^x = 0 \end{cases} \Leftrightarrow \begin{cases} y^2 - x^2 - 2x = 0 \\ y = 0 \end{cases}$$

Since  $e^x > 0 \forall x$ .

$$y = 0 \Rightarrow -x^2 - 2x = 0 \\ \Rightarrow x = 0 \text{ or } x = -2$$

$$\begin{matrix} (0,0) \\ (-2,0) \end{matrix} \left. \vphantom{\begin{matrix} (0,0) \\ (-2,0) \end{matrix}} \right\} \text{c.p.}$$

$$H_f(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{matrix} \Delta_1 = -2 \\ \Delta_2 = -4 < 0 \end{matrix} \Rightarrow \text{saddle at } (0,0)$$

$$H_f(-2,0) = \begin{bmatrix} e^{-2}(-4 + 8 - 2) & 0 \\ 0 & e^{-2} \end{bmatrix} \quad \begin{matrix} \Delta_1 = 2e^{-2} > 0 \\ \Delta_2 = 2e^{-4} > 0 \end{matrix} \\ \Rightarrow \text{local min at } (-2,0).$$

**Problem 2.** Find the absolute maximum and minimum values of  $f(x,y) = x^2 + 2y^2 - 2x$  on the closed disk  $\{(x,y) \mid x^2 + y^2 \leq 9\}$ . State where the maximum and minimum values are attained.

$$\nabla f = (2x-2, 4y)$$

$$\nabla f = 0 \iff \begin{cases} 2x-2=0 \\ 4y=0 \end{cases}$$

$\implies (1,0)$  interior pt.

$$g = x^2 + y^2 = 9$$

$$\nabla g = (2x, 2y)$$

L.M.  
 $\nabla f = \nabla g \cdot \lambda$

$$\left\{ \begin{array}{l} 2x-2 = 2\lambda x \quad \textcircled{1} \\ 4y = 2\lambda y \quad \textcircled{2} \\ x^2 + y^2 = 9 \quad \textcircled{3} \end{array} \right.$$

$$\begin{aligned} \textcircled{2} &\implies 0 = 4y - 2\lambda y = 2y(2-\lambda) \\ &\implies y=0 \quad \text{OR} \quad \lambda=2 \end{aligned}$$

Case 1  $y=0$   
 $\textcircled{3} \implies x = \pm 3$

OR

Case 2  $\lambda=2$   
 $2x-2 = 4x$   
 $-2 = 2x$   
 $-1 = x$

$$\textcircled{3} \implies y = \pm\sqrt{8}$$

	$x^2 + 2y^2 - 2x$	
int. pt. $(1,0)$	$1 - 2 = -1$	min
$(3,0)$	$9 - 6 = 3$	
$(-3,0)$	$9 + 6 = 15$	
$(-1, \sqrt{8})$	$1 + 16 + 2 = 19$	} max
$(-1, -\sqrt{8})$	$1 + 16 + 2 = 19$	

Since  $\{(x,y) \mid x^2 + y^2 \leq 9\}$  is closed & bounded,  $f$  must have a max and a min value on it.

Max value 19 on  $(-1, \pm\sqrt{8})$

Min value -1 on  $(1,0)$ .

**Problem 3.** Let  $\mathbf{x}(t) = 2\cos t \mathbf{i} - 3\sin t \mathbf{j}$ , for  $0 \leq t \leq \pi$ .

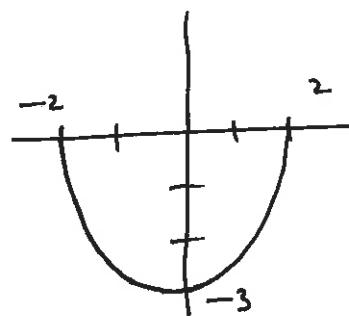
a. (i) Sketch the image of the path  $\mathbf{x}(t)$ , using arrows to indicate its direction.

$$\vec{\mathbf{x}}(t) = (2\cos t, -3\sin t)$$

$$\vec{\mathbf{x}}(0) = (2, 0)$$

$$\vec{\mathbf{x}}\left(\frac{\pi}{2}\right) = (0, -3)$$

$$\vec{\mathbf{x}}(\pi) = (-2, 0)$$



ii. Find a parametric equation for the tangent line to  $\mathbf{x}(t)$  when  $t = \frac{\pi}{6}$ .

$$\vec{\mathbf{x}}\left(\frac{\pi}{6}\right) = \left(2\cos\frac{\pi}{6}, -3\sin\frac{\pi}{6}\right) = \left(\frac{2\sqrt{3}}{2}, -3 \cdot \frac{1}{2}\right) = \left(\sqrt{3}, -\frac{3}{2}\right)$$

$$\vec{\mathbf{x}}'(t) = (-2\sin t, -3\cos t)$$

$$\vec{\mathbf{x}}'\left(\frac{\pi}{6}\right) = \left(-2 \cdot \frac{1}{2}, -3 \cdot \frac{\sqrt{3}}{2}\right) = \left(-1, -\frac{3\sqrt{3}}{2}\right)$$

$$\ell(s) = \left(\sqrt{3}, -\frac{3}{2}\right) + s \left(-1, -\frac{3\sqrt{3}}{2}\right)$$

b. (i) Why does the equation  $xz^4 + y^3z + x^5y = 3$  implicitly define  $z$  as a function of  $x$  and  $y$  near the point  $(x, y, z) = (1, 1, 1) = \mathbf{p}$ ?

(ii) If  $g(x, y) = z$  is the solution function near  $\mathbf{p}$ , what is  $Dg$  at  $\mathbf{p}$ ? Compute the derivative matrix  $Dg$  near  $\mathbf{p}$  in terms of  $x, y$ , and  $z$ .

$$\text{Let } F = xz^4 + y^3z + x^5y : \mathbb{R}^3 \rightarrow \mathbb{R}.$$

$$DF = \begin{bmatrix} z^4 + 5x^4y & 3y^2z + x^5 & 4xz^3 + y^3 \end{bmatrix}$$

$$DF(1, 1, 1) = \begin{bmatrix} 6 & 4 & 5 \end{bmatrix} \quad \text{want to solve for } z$$

$\det [5] = 5 \neq 0$ , hence the  $\exists g(x, y) = z$  solving  $z$  near  $(1, 1, 1)$

$$Dg(1, 1, 1) = -[5]^{-1} [6 \ 4] = -\left[\frac{1}{5}\right] [6 \ 4] = \left[-\frac{6}{5} \quad -\frac{4}{5}\right]$$

$$\begin{aligned} Dg(x, y, z) &= -[4xz^3 + y^3]^{-1} [z^4 + 5x^4y \quad 3y^2z + x^5] \\ &= \left[ -\frac{z^4 + 5x^4y}{4xz^3 + y^3} \quad -\frac{3y^2z + x^5}{4xz^3 + y^3} \right] \end{aligned}$$

**Problem 4.**

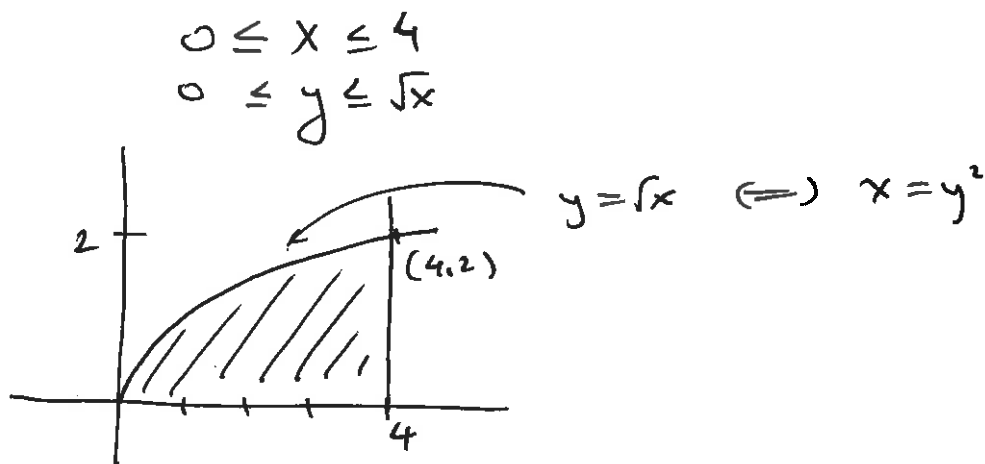
Consider the integral  $\int_0^4 \int_0^{\sqrt{x}} 2y \cos(x^2) dy dx$ .

a. Evaluate this integral directly.

$$\begin{aligned} &= \int_0^4 \int_0^{\sqrt{x}} 2y \cos(x^2) dy dx = \int_0^4 y^2 \cos x^2 \Big|_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^4 \left( (\sqrt{x})^2 \cos x^2 - 0 \right) dx = \int_0^4 x \cos x^2 dx \\ &= \int_0^{16} \frac{1}{2} du \cos u = \frac{1}{2} \sin u \Big|_0^{16} = \frac{1}{2} (\sin 16 - \sin 0) \end{aligned}$$

$u = x^2$   
 $du = 2x dx$

b. Sketch the domain of integration.



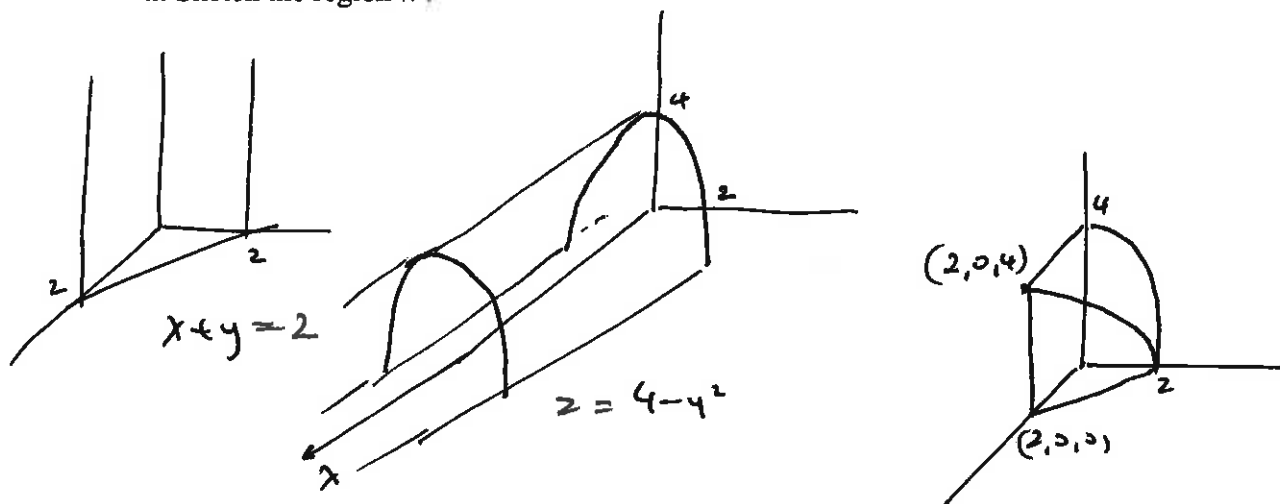
c. Reverse the order of integration of the integral above.

$$\int_0^2 \int_{y^2}^4 2y \cos(x^2) dx dy$$

$$\begin{aligned} 0 &\leq y \leq 2 \\ y^2 &\leq x \leq 4 \end{aligned}$$

**Problem 5.** Let  $W$  be the region in the first octant of  $\mathbb{R}^3$  ( $x, y, z \geq 0$ ), bounded by the cylinder  $z = 4 - y^2$ , and the planes  $x + y = 2$ ,  $x = 0$ ,  $y = 0$ , and  $z = 0$ .

a. Sketch the region  $W$ .



b. Calculate  $\iiint_W y^2 dV$ .

$$\begin{aligned} \int_0^2 \int_0^{2-x} \int_0^{4-y^2} y^2 dz dy dx &= \int_0^2 \int_0^{2-x} y^2 z \Big|_{z=0}^{z=4-y^2} dy dx \\ &= \int_0^2 \int_0^{2-x} y^2(4-y^2) dy dx = \int_0^2 \int_0^{2-x} (4y^2 - y^4) dy dx \\ &= \int_0^2 \left. \frac{4y^3}{3} - \frac{y^5}{5} \right|_0^{2-x} dx = \int_0^2 \left( \frac{4}{3}(2-x)^3 - \frac{1}{5}(2-x)^5 \right) dx \\ &\quad \text{let } u=2-x \\ &\quad \quad du=-dx \\ &= - \int_2^0 \left( \frac{4}{3} u^3 - \frac{1}{5} u^5 \right) du = \left. \frac{1}{3} u^4 - \frac{1}{30} u^6 \right|_0^2 = \frac{16}{3} - \frac{64}{30} = \frac{16}{5} \end{aligned}$$

(PTO) for other orders:

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} y^2 dz dx dy$$

$$= \int_0^2 \int_0^{2-y} y^2(4-y^2) dx dy$$

$$= \int_0^2 \underbrace{y^2(4-y^2)(2-y)}_{8-4y-2y^2+y^3} dy$$

$$= \int_0^2 (8y^2 - 4y^3 - 2y^4 + y^5) dy$$

$$= \left. \frac{8}{3}y^3 - y^4 - \frac{2}{5}y^5 + \frac{y^6}{6} \right|_0^2 = \frac{64}{3} - 16 - \frac{64}{5} + \frac{64}{3}$$

$$= \frac{96}{3} - 16 - \frac{64}{5} = 16 - \frac{64}{5}$$

$$= \frac{80-64}{5} = \frac{16}{5}$$

OR

$$\int_0^2 \int_0^{4-y^2} \int_0^{2-y} y^2 dx dz dy$$

$$= \int_0^2 \int_0^{4-y^2} y^2(2-y) dz dy$$

$$= \int_0^2 y^2(2-y)(4-y^2) dy$$

$$= \int_0^2 (8y^2 - 4y^3 - 2y^4 + y^5) dy = \dots \stackrel{\text{as above}}{=} \frac{16}{5}$$