2006

1. **a**. Complete the following ε-N definition (Definition 16.2) for a convergent sequence:

Definition: A sequence (s_n) is said to converge to the real number L provided that......

b. Prove that if (s_n) is a convergent sequence and $k \in \mathbf{R}$, then

$$\lim_{n\to\infty} (ks_n) = k\left(\lim_{n\to\infty} s_n\right).$$

You are asked to prove Theorem 17.1(b). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 17.1(a-d). Simply, it does **not** suffice to say "This follows Theorem 17.1". You are expected to provide a detailed proof.

2. **a**. Calculate $\lim_{n\to\infty} \frac{6n}{3n+1}$ and justify your answer by using limit theorems.

b. Let A be your answer to part (a). By using the ε-N definition of convergence, prove that

$$\lim_{n\to\infty} \frac{6n}{3n+1} = A.$$

3. A sequence (s_n) is defined recursively by

$$s_1 = 2$$
$$s_{n+1} = \frac{5+2s_n}{9}, \text{ for } n \geq 1$$

a. Prove that (s_n) is a bounded and monotone decreasing sequence.

b. Explain why (s_n) is a convergent sequence and find $\lim_{n\to\infty} s_n$.

4. **a**. State the *Completeness Axiom for* **R**.

b. Let S be a nonempty and bounded subset of **R**, and define $kS = \{kx : x \in S\}$ for $k \in \mathbf{R}$. Prove that $\sup(kS) = k\sup(S)$, if $k > 0$.

5. **SHOW NO WORK and NO PARTIAL CREDIT- TRUE OR FALSE**. **CIRCLE YOUR ANSWERS**.

Correct answers are +5 points each, wrong answers are -1 point each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5b will be added to your total grade only if it is positive.

TRUE FALSE **i**) $\forall x \in \mathbf{R}\ \forall y \in \mathbf{R}\ (y > 0 \Rightarrow \exists n \in \mathbf{N}$ such that $nx \geq y)$

TRUE FALSE **ii**) For every given pair of real numbers x and y with $x < y$, there are integers m and n such that $x < \frac{m}{n}\pi < y$.

TRUE FALSE **iii**) Let (s_n) be a sequence of real numbers. Then, $\lim_{n\to\infty} s_n = +\infty$ if and only if $\lim_{n\to\infty} (s_n)^2 = +\infty$.

TRUE FALSE **iv**) Every Cauchy sequence in **R** is convergent.

2005

1. **a**. Complete the following ε-N definition (Definition 16.2) for a convergent sequence:

Definition: A sequence (s_n) is said to converge to the real number L provided that..........

b. Prove that if (s_n) and (t_n) are convergent sequences, then

$$\lim_{n\to\infty}(s_n+t_n) = \left(\lim_{n\to\infty} s_n\right) + \left(\lim_{n\to\infty} t_n\right).$$

You are asked to prove Theorem 17.1(a). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 17.1(a). Simply, it does **not** suffice to say “This follows Theorem 17.1(a)”. You are expected to provide a detailed proof.

2. By using the ε-N definition of convergence, prove that

$$\lim_{n\to\infty}\frac{n+3}{n^2-2} = 0.$$

3. **a**. State the *Completeness axiom for* **R**.

b. Let S be a nonempty and bounded subset of **R**, and define $-S = \{-x : x \in S\}$. Prove that $\inf(-S) = -\sup(S)$.

4. A sequence (s_n) is defined recursively by

$$s_1 = 0, \text{ and } s_{n+1} = \sqrt{6+s_n}, \text{ for } n \geq 1$$

a. Prove that (s_n) is a bounded and monotone increasing sequence.

b. Explain why (s_n) is a convergent sequence and find $\lim_{n\to\infty} s_n$.

c. EXRTA CREDIT (5 points) Write your answer on the back of this page.

For this part of the question, one starts with a different value $k = s_1$ and keeps the same recursive definition: $s_{n+1} = \sqrt{6+s_n}$, for $n \geq 1$.

Determine for what starting values of k this sequence is monotone increasing and for what starting values of k this sequence is monotone decreasing? In either case, what are the possibilities for the limit? Justify your answers with few sentences, but not with a formal proof.

5. **TRUE OR FALSE**.

TRUE FALSE **a**. Sequences (s_n) and (t_n) converge if and only if the sequence (s_nt_n) converges.

TRUE FALSE **b**. For every real number L, there exists a sequence (s_n) of irrational numbers such that $\lim_{n\to\infty} s_n = L$.

TRUE FALSE **c**. Let (s_n) be a sequence of nonzero real numbers. Then, $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} |\frac{1}{s_n}| = +\infty$

TRUE FALSE **d**. Every monotone bounded sequence in **R** is Cauchy.

TRUE FALSE **e**. If a sequence (s_n) diverges in **R**, then $\exists\varepsilon > 0 \ni \forall M, \exists m,n \in \mathbf{N} \ni m,n > M$ and $|s_n - s_m| \geq \varepsilon$.

2004

1. Give the definition of a sequence $\{a_n\}$ diverging to $+\infty$, and by using this definition prove the following (Theorem 2.3.3.c):

If a sequence $\{a_n\}$ diverges to $+\infty$, then $\{ca_n\}$ diverges to $+\infty$ for all positive constants c.

2. **a**. Complete the following ε-n^* definition for a convergent sequence:

Definition:

A sequence $\{a_n\}$ is said to converge to the real number A provided that .....

b. Calculate $\lim_{n\to\infty} \frac{3n+1}{4n+2}$ and justify your answer by using limit theorems.

c. Let A be your answer to part (b). By using the ε-N definition of convergence, prove that

$$\lim_{n\to\infty} \frac{3n+1}{4n+2} = A.$$

3. A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1$$
$$a_{n+1} = \sqrt{6a_n + 3}, \text{ for } n \geq 1$$

a. Prove that $\{a_n\}$ is a bounded and monotone increasing sequence.

b. Explain why $\{a_n\}$ is a convergent sequence, by explicitly stating the theorem(s) that you use.

c. Find $\lim_{n\to\infty} a_n$.

4. **a**. State the *Bolzano-Weierstrass Theorem*. (Either for sets or for subsequences, but not both).

b. Give the definition of a *Cauchy* sequence.

c. Prove that every convergent sequence is Cauchy.

5. **TRUE OR FALSE**.

TRUE FALSE **a**. If both $\lim_{n\to\infty}(a_nb_n)$ and $\lim_{n\to\infty} a_n$ exist and are finite, then $\lim_{n\to\infty} b_n$ exists.

TRUE FALSE **b**. $\lim_{n\to\infty} \frac{(\cos n)^n}{\sqrt{n}}$ exists.

TRUE FALSE **c**. $\lim_{n\to\infty} \frac{100^n}{n!} = 0$.

TRUE FALSE **d**. The set of accumulation points of $[0, \sqrt{2}) \cap \mathbf{Q}$ in $\mathbf{R}$ is $[0, \sqrt{2})$.

TRUE FALSE **e**. A sequence $\{a_n\}$ diverges if and only if it has a divergent subsequence.

2003

1. **a**. Complete the following ε-N definition for a convergent sequence:

Definition:

A sequence (s_n) is said to converge to the real number L provided that .....

b. By using the ε-N definition of convergence, prove that $\lim_{n\to\infty} \frac{4n+1}{2n+2} = 2$.

2. Prove that if (s_n) and (t_n) are convergent sequences, then

$$\lim_{n\to\infty} (s_n + t_n) = \left(\lim_{n\to\infty} s_n\right) + \left(\lim_{n\to\infty} t_n\right).$$

3. A sequence (s_n) is defined recursively by

$$s_1 = 0$$

$$s_{n+1} = \sqrt{s_n + 1}\,, \text{ for } n \geq 1$$

a. Prove that (s_n) is a bounded and monotone increasing sequence.

b. Explain why (s_n) is a convergent sequence and find $\lim_{n\to\infty} s_n$.

4. Let $S = \{r \in \mathbf{Q} : r^2 \leq 4\}$ be a subset of **R**. In parts (**a**) and (**b**), prove your assertions by using definitions and theorems. State or indicate the theorems you use.

a. Find the interior of S and prove your assertion.

b. Find the supremum of S and prove your assertion.

c. No need for explanations or proofs for this part. Circle your answer or write in your answer. If an answer is ”Does not exist”, then state it so.

Is S an open set? Yes or No | Is S a closed set? Yes or No

The closure of S is ________. | The boundary of S is_________.

The infimum of S is________. | The maximum of S is________.

5. **TRUE OR FALSE**.

TRUE FALSE **a**. Every bounded subset of **R** has a largest (maximum) element.

TRUE FALSE **b**. For every given pair of real numbers x and y with $0 < x < y$, there are integers m and n such that $x < \sqrt{\frac{m}{n}} < y$.

TRUE FALSE **c**. $\forall x \in \mathbf{R}\ \forall y \in \mathbf{R}\ \exists n \in \mathbf{N}$ such that $nx \geq y$.

TRUE FALSE **d**. An open subset S of **R** can not be a closed subset of **R**.

TRUE FALSE **e**. If a sequence (s_n) in **R** satisfies the inequality $|s_n - s_m| \leq \frac{1}{n}$ for every natural number m and n with $n < m$, then (s_n) must converge to a limit in **R**.

2002

1. Prove that if (s_n) is a convergent sequence, then $\lim_{n\to\infty}(ks_n) = k\lim_{n\to\infty}(s_n)$ for any $k \in \mathbf{R}$.

2. A sequence (s_n) is defined by

$$s_1 = 1$$
$$s_{n+1} = \sqrt{2s_n + 15}, \text{ for } n \geq 1$$

a. Prove that (s_n) is a bounded and monotone sequence.
b. Explain why (s_n) is a convergent sequence.
c. Find $\lim_{n\to\infty} s_n$.

3. Let $S = [1,2)$. In parts (**a**) and (**b**), prove your assertions by using definitions and theorems. State or indicate the theorems you use.
a. Find the interior of S and prove your assertion.
b. Is 2 an accumulation point of S? Prove your assertion.
c. No need for explanations for this part. Circle your answer or write in your answer.
Is S an open set? Yes or No
Is S a closed set? Yes or No
The closure of S is ________.
The boundary of S is_________.

4. **a**. State the following.
The Completeness Axiom for R (**The Least Upper Bound property of R**):
The Bolzano-Weierstrass Theorem:
b. Prove Bolzano-Weierstrass Theorem by starting from The Completeness Axiom.

5. **TRUE OR FALSE**.
TRUE FALSE **a**. The set $\{(-1)^n(1-\frac{1}{n}) : n \in \mathbf{N}\}$ does not have a supremum.
TRUE FALSE **b**. By the completeness axiom, every bounded subset of **R** has a largest (maximum) element.
TRUE FALSE **c**. $\forall x > 0$ and $x \in \mathbf{R}$ $\forall y \in \mathbf{R}$ $\exists n \in \mathbf{N}$ such that $nx \geq y$.
TRUE FALSE **d**. For every given pair of real numbers x and y with $x < y$, there are integers m and n such that $x < \frac{\sqrt{m}}{n} < y$.
TRUE FALSE **e**. Every Cauchy sequence in **Q** converges to a limit in **R**.

SOME PROBLEMS FROM OLD MIDTERMS related to our material:

2002 #4. **a**. Complete the following definition.

Definition:

A sequence (s_n) is said to converge to the real number L provided that .....

b. By using the definition of convergence, prove that

$$\lim_{n\to\infty} \frac{3n+2}{2n+3} = \frac{3}{2}.$$

2004 #2. **a**. State *The Principle of Mathematical Induction.*

b. Let a_n (for $n \in \mathbf{N}$) be defined recursively by $a_1 = 2$ and $a_{n+1} = \sqrt{4a_n - 3}$ for $n \geq 1$. Prove the following by using *The Principle of Mathematical Induction.*

i. For all $n \in \mathbf{N}$, $a_n \leq 3$ and

ii. For all $n \in \mathbf{N}$, $a_n \leq a_{n+1}$.

4. **a**. State *The Completeness Axiom for* **R**.

b. Let S be non-empty subset of **R** and k be an upper bound of S.

Prove that k is the least upper bound of S if and only if for every $\varepsilon > 0$ there exists $s \in S$ such that $k - \varepsilon < s$.