22M:55-MIDTERM 1—-2006

1. **a**. Prove that if $\frac{x}{x-2} \leq 3,$ then $x < 2$ or $x \geq 3,$ for $x \in \mathbf{R}$.

b. Prove that if $\frac{u-2}{u} \geq 2,$ then $u < 0,$ for $u \in \mathbf{R}$.

2. Prove that if A and B are sets, then $A \setminus (A \setminus B) = A \cap B$.

3. Let $f : A \to B$ be an arbitrary function, and $C \subset A$ and $D \subset B$ be arbitrary subsets.

a. Prove that $f^{-1}\left(B \setminus D\right) = A \setminus f^{-1}(D)$

You are asked to prove Theorem 7.15(g). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 7.15(g) Simply, it does **not** suffice to say "This follows Theorem 7.15(g)". You are expected to provide a proof.

b. Given the statement

"$f\left(A \setminus C\right) = B \setminus f(C),$ for every function $f : A \to B,$ and every subset $C \subset A$",

(i) State whether this statement is true or false, and (ii) Prove the statement if it is true, or give a counterexample (with explicit f, A, B & C) if it is false.

4. **a**. (10 points) Prove the inequality $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|,$ for real numbers $x_1, x_2, \ldots, x_n,$ by using *The Principle of Mathematical Induction*, and assuming the standard Triangle Inequality: $\forall x, y \in \mathbf{R}, |x+y| \leq |x| + |y|$.

b. (10 points) By using *The Principle of Mathematical Induction*, prove that $7^n - 3^n$ is a multiple of 4.

(**EXTRA CREDIT - 10 points**) **Provide a proof of** *The Principle of Mathematical Induction* by assuming the Well-Ordering property of $\mathbf{N},$ on the back of this page. Of course, you should state the theorem, also.

5. **TRUE OR FALSE**

CIRCLE YOUR ANSWERS. NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK. **HINT**: **READ VERY CAREFULLY**.

Correct answers are +4 points each, wrong answers are -1 point each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5 will be added to your total grade only if it is positive.

TRUE FALSE

a. The negation of the statement $\exists x \in \mathbf{R},\ \forall y \in \mathbf{R} \ni (x+y \leq 1,$ or $0 \geq xy,$ or $xy > 1)$ is equivalent to $\forall x \in \mathbf{R} \ni \exists y \in \mathbf{R}\ (x+y > 1 \Rightarrow 0 < xy \leq 1)$.

TRUE FALSE **b**. Both of the following statements (i) and (ii) are true:

$i. \forall x \in \mathbf{R},\ \forall y \in \mathbf{R}, \exists z \in \mathbf{R} \ni (x+z = y)$ and

$ii. \forall x \in \mathbf{R},\ \forall y \in \mathbf{R}, \exists z \in \mathbf{R} \ni (x+z \neq y)$.

TRUE FALSE **c**. The set $X = \left\{\sqrt{\frac{p}{q}} : \frac{p}{q} \in \mathbf{Q} \text{ and } \frac{p}{q} > 0\right\}$ is an uncountable set, since it contains infinitely many irrational numbers.

TRUE FALSE **d**. The relation S on $\mathbf{N}$ defined below is an equivalence relation: $xSy \Leftrightarrow x+y$ is an even number.

TRUE FALSE **e**. In order to prove "$\forall n \in A \ni p(n)$" is false for a given set $A,$ we must show that $p(n)$ is false for all n in A.

22M:**55-MIDTERM 1—-2005**

1. Assume that x and y real numbers. For each of the following statements,
(i) State whether it is true or false, and
(ii) Prove the statement if it is true, or give a counterexample if it is false.
a. If x is irrational and y is irrational, then xy is irrational.
b. If xy is irrational, then x is irrational or y is irrational.

2. Prove that if $C = A \cup B$ and $A \cap B = \emptyset,$ then $A = C \setminus B$.

3.You are asked to prove Theorems 7.14(a) and 7.17(a). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 7.14(b) or 7.17(a). Simply, it does **not** suffice to say “This follows Theorem 7.14(b) and 7.17(a)”. You are expected to provide proofs.
a. Let $f : A \to B,$ and $C \subset A.$ Prove that $C \subset f^{-1}[f(C)].$
b. Prove that if f is injective then $C = f^{-1}[f(C)].$

4. **a**. State *The Principle of Mathematical Induction.*
b. Indicate for which natural numbers n the inequality $n^2 \leq n!$ is true.
Prove $n^2 \leq n!$ for the numbers you have indicated by using *The Principle of Mathematical Induction.*

5. **TRUE OR FALSE**
CIRCLE YOUR ANSWERS. NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each, wrong answers are -1 point each, ambiguous answers are -2 points each, and no answers are 0 point each. Total of problem 5 will be added to your total grade only if it is positive.**HINT**: **Read very carefully**.
TRUE FALSE **a**. The negation of the statement $\forall x \in \mathbf{R}, \exists y \in \mathbf{R} \ni (x + y > 1$ and $xy > 0)$ is equivalent to $\exists x \in \mathbf{R} \ni \forall y \in \mathbf{R}\ (x + y > 1 \Rightarrow xy \leq 0)$.
TRUE FALSE **b**. There exists no bijection between the set of rational numbers **Q** and the set of real numbers **R**.
TRUE FALSE **c**. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R} \ni \forall z \in \mathbf{R}\ (xz = y)$.
TRUE FALSE **d**. The relation S on **R** defined below is an equivalence relation: $xSy \Leftrightarrow x - y$ is a rational number.
TRUE FALSE **e**. For a set A with more than one element, it takes only one counterexample to prove “$\exists n \in A \ni p(n)$" is false.

22M:55-MIDTERM 1—-2004

1. Let $f : A \to B$ and $g : B \to C$ be functions.
a. Prove that if f and g are injective functions then $g \circ f$ is an injective function.
b. Prove that if f and g are surjective functions then $g \circ f$ is a surjective function.
c. Prove that if f and g are bijective functions then $g \circ f$ is invertible.

2. **a**. State *The Principle of Mathematical Induction.*
b. Let a_n (for $n \in \mathbf{N}$) be defined recursively by $a_1 = 2$ and $a_{n+1} = \sqrt{4a_n - 3}$ for $n \geq 1$. Prove the following by using *The Principle of Mathematical Induction.*
i. For all $n \in \mathbf{N},\ a_n \leq 3$ and
ii. For all $n \in \mathbf{N},\ a_n \leq a_{n+1}$.

3. Prove that if $\frac{x-1}{x} \leq 2,$ then $x > 0$ or $x \leq -1,$ for $x \in \mathbf{R}$.

4. **a**. State *The Completeness Axiom for* **R**.
b. Let S be non-empty subset of **R** and k be an upper bound of S.
Prove that k is the least upper bound of S if and only if for every $\varepsilon > 0$ there exists $s \in S$ such that $k - \varepsilon < s$.

5. **TRUE OR FALSE**. CIRCLE YOUR ANSWERS. NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each, wrong answers are -1 point each, ambiguous answers are -2 points each, andno answers are 0 point each. Total of problem 5 will be added to your total grade only if it is positive. **HINT**: **Read very carefully**.
TRUE FALSE **a**. The negation of the statement $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R}\ (xy > 1 \Rightarrow x > y)$ is $\exists x \in \mathbf{R}\ \forall y \in \mathbf{R}\ (x \leq y \Rightarrow xy \leq 1)$.
TRUE FALSE **b**. The set of rational numbers **Q** is equipotent to the set of irrational numbers $\mathbf{R} - \mathbf{Q}$, that is $\mathbf{Q} \sim (\mathbf{R} - \mathbf{Q})$.
TRUE FALSE **c**. $\forall x, y \in \mathbf{R},\ \exists n \in \mathbf{N}$ such that $nx > y$.
TRUE FALSE **d**. There are sets A, B and C such that $(A - B) \cup C = (A \cup C) - B$.
TRUE FALSE **e**. There exists a smallest positive real number.
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22M:55-MIDTERM 1—-2003

1. Prove that if $\frac{x+2}{x-2} \leq 5,$ then $x < 2$ or $x \geq 3,$ for $x \in \mathbf{R}$.

2. Prove that $A \cap B$ and $A \setminus B$ are disjoint and $A = (A \cap B) \cup (A \setminus B),$ for any two sets A and B.

3. Let $f : A \to B,$ and $D \subset B.$ Prove that $f[f^{-1}(D)] \subset D.$

You are asked to prove Theorem 7.14(b). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 7.14(b). Simply, it does **not** suffice to say "This follows Theorem 7.14(b)". You are expected to provide a proof.

b. Provide a counterexample to show that $D \subset f[f^{-1}(D)]$ is not always correct.

4. **a**. State *The Principle of Mathematical Induction.*

b. Indicate for which natural numbers n, $2^n \leq n!$ is true. Prove this inequality with your indication by using *The Principle of Mathematical Induction.*

5. **TRUE OR FALSE**. CIRCLE YOUR ANSWERS. SHOW NO WORK.

Correct answers are +4 points each, wrong or ambiguous answers are -2 points each, and no answers are 0 point each. Total of problem 5 will be added to your total grade only if it is positive.

TRUE FALSE **a**. The negation of the statement $\exists x \in \mathbf{R} \ni \forall y \in \mathbf{R}\ (x \leq y \Rightarrow xy > 1)$ is $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R} \ni (x > y$ and $xy \leq 1)$.

TRUE FALSE **b**. The set of integers $\mathbf{Z}$ is a countable set, although one can never finish counting them.

TRUE FALSE **c**. $\exists z \in \mathbf{R} \ni \left([\forall x,y \in \mathbf{R}(x < y \Rightarrow xz < yz)] \text{ is false.}\right)$

TRUE FALSE **d**. The relation S on $\mathbf{R}$ defined below is an equivalence relation: $xSy \Leftrightarrow xy \geq 0$.

TRUE FALSE **e**. There exists a smallest positive real number.

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22M:55-MIDTERM 1—-2002

1. Prove that if $\frac{x+1}{x-1} \leq 3,$ then $x < 1$ or $x \geq 2.$

2. By using *The Principle of Mathematical Induction*, prove that $9^n - 4^n$ is a multiple of 5.

3. Let $f : A \to B,$ and D_1 and D_2 be subsets of $B.$ Prove that

$$f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2).$$

You are asked to prove Theorem 7.14(e). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 7.14(e). Simply, it does **not** suffice to say ”This follows Theorem 7.14(e)”. You are expected to provide a proof.

4. **a**. Complete the following definition.

Definition:

A sequence (s_n) is said to converge to the real number L provided that .....

b. By using the definition of convergence, prove that

$$\lim_{n\to\infty} \frac{3n+2}{2n+3} = \frac{3}{2}.$$

5. **TRUE OR FALSE**. CIRCLE YOUR ANSWERS. SHOW NO WORK.

Correct answers are +4 points each, wrong or ambiguous answers are -1 point each, and no answers are 0 point each. Total of problem 5 will be added to your total grade only if it is positive.

TRUE FALSE **a**. The negation of the statement $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R} \ni (x > y \Rightarrow xy \leq 1)$ is $\exists x \in \mathbf{R} \ni \forall y \in \mathbf{R}(x \leq y$ and $xy > 1).$

TRUE FALSE **b**. The set of Irrational numbers is not a countable set.

TRUE FALSE **c.** $\forall x,y,z \in \mathbf{R}($if $x < y,$ then $xz < yz)$

TRUE FALSE **d**. The relation S on $\mathbf{R}$ defined below is not an equivalence relation: $xSy \Leftrightarrow |x-y| \leq 1.$

TRUE FALSE **e**. For sets A, B and C,

$(A \cap B = \emptyset$ and $B \cap C = \emptyset$ and $A \cap C = \emptyset)$ if and only if $A \cap B \cap C = \emptyset.$