1. a. Complete the following ε-N definition (Definition 4.1.2) for a convergent sequence:
Definition:
A sequence (s_n) is said to converge to the real number L provided that....

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \in \mathbb{N} \ (n \geq N \Rightarrow |s_n - L| < \varepsilon)$

b. By using **only** the ε-N definition of convergence and inequalities, prove that

$$\lim_{n\to\infty} \frac{3n-1}{5n+4} = \frac{3}{5}.$$

(Using limit theorems (4.2) will not earn credit.)

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$ (and hence $\frac{1}{N} < \varepsilon$)

$\forall n \geq N$,

$$|s_n - L| = \left|\frac{3n-1}{5n+4} - \frac{3}{5}\right| = \left|\frac{15n - 5 - 15n - 12}{5(5n+4)}\right| = \left|\frac{-17}{5(5n+4)}\right|$$

$$= \frac{17}{5(5n+4)} \leq \frac{17}{25n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

2. A sequence (s_n) is defined by

$$s_1 = 0$$
$$s_{n+1} = \sqrt{3 + s_n}\,, \text{ for } n \geq 1.$$

a. Prove that (s_n) is a bounded and monotone sequence.

To prove $p(n)$ $\quad 0 \leq s_n \leq 6 \quad \forall n \in \mathbb{N}$.

Proof by induction.

$P(1)$, base case $\quad 0 \leq s_1 = 0 \leq 6$.

Assume $p(k)$ to prove $p(k+1)$.

$p(k)$ $\quad 0 \leq s_k \leq 6$

$3 \leq s_k + 3 \leq 9$

$0 \leq \sqrt{3} \leq \sqrt{s_k + 3} = s_{k+1} \leq \sqrt{9} \leq 6$. So $\quad 0 \leq s_{k+1} \leq 6 \quad p(k+1)$

Hence Thm of induction: $\forall n \in \mathbb{N}$, $\quad 0 \leq s_n \leq 6$

To prove $q(n)$ $\quad s_n \leq s_{n+1}$. Proof by induction.

$q(1)$ $\quad 0 = s_1 \leq s_2 = \sqrt{3}$, base case.

Assume $q(k)$ to prove $q(k+1)$.

$s_k \leq s_{k+1} \quad (q(k))$

$s_k + 3 \leq s_{k+1} + 3$

$s_{k+1} = \sqrt{s_k + 3} \leq \sqrt{s_{k+1} + 3} = s_{k+2}$

$s_{k+1} \leq s_{k+2} \quad (q(k+1))$

Theorem of induction $\Rightarrow \forall n \in \mathbb{N}$. $s_n \leq s_{n+1}$.

b. Explain why (s_n) is a convergent sequence. If you use a theorem, give a statement of the whole theorem (do not just give a number).

Thm. Every bounded monotone sequence in $\mathbb{R}$ is convergent in $\mathbb{R}$.

(s_n) is bounded & monotone $\Rightarrow \lim_{n \to \infty} s_n = L \in \mathbb{R}$.

c. Find $\lim_{n\to\infty} s_n$.

$\lim s_{n+1} = \lim s_n = L$. $\quad L \geq 0$ since $s_n \geq 0$. Thm 4.2.5

$s_{n+1} = \sqrt{3 + s_n}$ $\qquad \lim_{n\to\infty} s_{n+1}^2 = \lim_{n\to\infty} 3 + s_n$

$s_{n+1}^2 = 3 + s_n$. $\qquad L^2 = 3 + L$

$L^2 - 3 - L = 0 \quad \Rightarrow \quad L = \dfrac{1 + \sqrt{13}}{2}$.

Version I

3. Consider $S = \{0\} \cup [1,3) \cup (3,4)$ as a subset of $\mathbb{R}$. Fill in your answers or circle your answers. You are not expected to provide justifications. No partial credit. No answers and wrong answers receive 0 points, no penalty in this question.

Leaving an answer blank will not earn credit. If an answer is the empty set, then state it so. If there exists no answer, then state it so, such as: *DNE (does not exist)*.

a. The supremum of S : $\sup S =$ $\underline{4}$

b. The infimum of S : $\inf S =$ $\underline{0}$

c. The maximum of S : $\max S =$ $\underline{\text{DNE}}$ since $4 \notin S$

d. The minimum of S : $\min S =$ $\underline{0}$

e. The interior of S : *int* $S =$ $(1,3) \cup (3,4)$

f. The boundary of S : *bd* $S =$ $\{0,1,3,4\}$

g. The accumulation points of S : $S' =$ $[1,4]$

h. The closure of S : *cl* $S =$ $\{0\} \cup [1,4]$.

Circle your answer.

i. Is S an open set? Yes or No

$\underset{\text{int } S}{\overset{(1,3)\cup(3,4)}{\shortparallel}} \neq S = \{0\} \cup [1,3) \cup (3,4)$

j. Is S a closed set? Yes or No

$bd\, S \not\subseteq S$ since $4 \in Bd(S)$, $4 \notin S$.

k. Is S a bounded set? Yes or No

bounded by 0 & 4.

Version II

3. Consider $S = (1,2) \cup (2,3] \cup \{5\}$ as a subset of $\mathbb{R}$. Fill in your answers or circle your answers. You are not expected to provide justifications. No partial credit. No answers and wrong answers receive 0 points, no penalty in this question.

Leaving an answer blank will not earn credit. If an answer is the empty set, then state it so. If there exists no answer, then state it so, such as: *DNE (does not exist)*.

a. The supremum of S : $\sup S =$ 5

b. The infimum of S : $\inf S =$ 1

c. The maximum of S : $\max S =$ 5

d. The minimum of S : $\min S =$ DNE since $1 \notin S$.

e. The interior of S : $int\ S =$ $(1,2) \cup (2,3)$

f. The boundary of S : $bd\ S =$ $\{1,2,3,5\}$

g. The accumulation points of S : $S' =$ $[1,3]$.

h. The closure of S : $cl\ S =$ $[1,3] \cup \{5\}$

Circle your answer.

i. Is S an open set? Yes or No $\quad$ $int\ S = (1,2) \cup (2,3) \neq S$

j. Is S a closed set? Yes or No $\quad$ $bd\ S \not\subseteq S$ since $1 \in bd(S)$, $1 \notin S$

k. Is S a bounded set? Yes or No

bounded by 1 & 5.

4. a. State the following.

The Completeness Axiom for $\mathbb{R}$ (The Least Upper Bound property of $\mathbb{R}$):

Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum (least upper bound) in $\mathbb{R}$.

The Density of $\mathbb{Q}$ in $\mathbb{R}$ (The density of rationals in real numbers)

For every real number x, y with $x<y$, there exists $r \in \mathbb{Q}$ s.t. $x < r < y$.

b. Prove the density of $\mathbb{Q}$ in $\mathbb{R}$. It suffices to prove this for positive numbers in this test. (You can continue on the back of the page or on the opposite left, if that is needed, but make sure to write **Problem 4** on top of it.) You are asked to prove Theorem 3.3.13. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 3.3.13 or its consequences. Simply, saying "This follows Theorem 3.3.13" will not earn any credit. You cannot use the result of any exercise unless you provide a solution of that exercise.

Proof as given in class. (next page: proof from the book.)

Given $0 < x < y$. $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{y-x} < n$ by Archimedean principle

* $\Rightarrow n(y-x) > 1$, $\frac{1}{n} < y-x$ and $ny > 0$.

Let $S = \{k \in \mathbb{N} \mid k \geq ny\}$. $S \neq \emptyset$ by Arch. Principle : $\mathbb{N}$ is not bounded.

Well ordering principle of $\mathbb{N}$ $\Rightarrow$ $\exists\, m = \min S \in \mathbb{N}$.

$m \in S \subseteq \mathbb{N}$, $m \geq ny$.

$m-1 \in \mathbb{Z}$.

$m-1 < ny \leq m$ since if $m \geq 2$, then $m-1 \in \mathbb{N}$, $m-1 \notin S$.

if $m = 1$, then $m-1=0$, and $0 < ny < 1$

Claim $nx < m-1$.

Suppose not : $nx \geq m-1$

$$\begin{array}{r} ny - nx > 1 \quad \text{* above} \\ + \quad \underline{} \\ ny > m, \end{array}$$ contradiction since $m \in S$.

Hence we have $nx < m-1 < ny < m$

$x < \frac{m-1}{n} < y$ $\quad m, n \in \mathbb{Z}$, $\frac{m-1}{n} \in \mathbb{Q}$. Take $r = \frac{m-1}{n}$

Alternative proof

4. a. State the following.
The Completeness Axiom for $\mathbb{R}$ (The Least Upper Bound property of $\mathbb{R}$):

The Density of $\mathbb{Q}$ in $\mathbb{R}$ (The density of rationals in real numbers)

b. Prove the density of $\mathbb{Q}$ in $\mathbb{R}$. It suffices to prove this for positive numbers in this test. (You can continue on the back of the page or on the opposite left, if that is needed, but make sure to write **Problem 4** on top of it.) You are asked to prove Theorem 3.3.13. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 3.3.13 or its consequences. Simply, saying "This follows Theorem 3.3.13" will not earn any credit. You cannot use the result of any exercise unless you provide a solution of that exercise.

Proof given in the book + Exercise 9.

Given $0 < x < y$. $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{y-x} < n$ by Archimedean Principle

Hence $1 < n(y-x) = ny - nx$

$1 + nx < ny$.

$nx > 0$.

Let $S = \{k \in \mathbb{N} \mid nx < k\}$. $S \neq \emptyset$ since $\mathbb{N}$ is unbounded

exc. 9 By Well-ordering principle $\exists\, m = \min S \in S \subseteq \mathbb{N}$.

Hence $nx < m$.

$m-1 \leq nx < m$ since if $m \geq 2$, then $m-1 \in \mathbb{N}$, $m-1 \notin S$

if $m = 1$, then $m-1 = 0 < nx$.

$$\left.\begin{array}{l} 1 + nx < ny \\ m-1 \leq nx \end{array}\right\} \Rightarrow \quad m \leq nx + 1 < ny$$

Combining all $\quad m-1 \leq nx < m < ny$

$x < \frac{m}{n} < y$, $\quad m, n \in \mathbb{Z}$,

$r = m/n \in \mathbb{Q}$.

Version I

5. TRUE or FALSE.

CIRCLE YOUR ANSWERS. SHOW NO WORK.

Correct answers are +4 points each, wrong answers are -1 points each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5 will be added to your total grade only if it is positive.

TRUE FALSE **a.** Let (s_n) be a sequence of real numbers.
If $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |s_n - L| < \varepsilon$, then the sequence (s_n) converges.

This means $\forall \varepsilon > 0\ \exists N = 1\ \forall n \in \mathbb{N},\ n \geq 1\ \ |s_n - L| < \varepsilon$

Hence s_n converges to L.

Actually $|s_n - L| < \varepsilon\ \forall \varepsilon > 0 \Rightarrow s_n = L$ for all n.

TRUE FALSE **b.** Let (s_n) be a sequence of nonzero real numbers.
Then, $\forall k \in \mathbb{R}$ ($\lim_{n\to\infty} s_n$ exists if and only if $\lim_{n\to\infty} ks_n$ exists).

$k = 0$ $\underbrace{\lim s_n \text{ exists} \Leftarrow \lim 0 \text{ exists}}_{\text{false}}$

TRUE FALSE **c.** Let (s_n) be a sequence of rational numbers, $\mathbb{Q}$.
If $\forall \varepsilon > 0, \exists M \ni \forall m,n \in \mathbb{N} \ni (m,n \geq M \Rightarrow |s_n - s_m| < \varepsilon)$,
then $\exists L \in \mathbb{Q}, \ni \forall \varepsilon > 0, \exists M \ni \forall n \in \mathbb{N}, (n \geq M \Rightarrow |s_n - L| < \varepsilon)$.

Cauchy
Convergent
to a limit
in $\mathbb{Q}$.

In $\mathbb{Q}$ Cauchy $\not\Rightarrow$ convergent

TRUE FALSE **d.** $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists n \in \mathbb{N}$ such that $((x > 0 \text{ and } y > 0) \Rightarrow x > \frac{y}{n})$.

$\Leftrightarrow \underbrace{\forall x \in \mathbb{R}, \forall y \in \mathbb{R}\ \exists n \in \mathbb{N}\ (x, y > 0\ \ nx > y)}_{\text{True by Archimedean Property}}$

TRUE FALSE **e.** Let (s_n) and (t_n) be sequences of real numbers such that $\forall n \in \mathbb{N}, s_n \leq t_n$.
If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

Thm. 4.2.12 p176

Version II

See the previous page for explanations.

5. **TRUE OR FALSE.**

CIRCLE YOUR ANSWERS. SHOW NO WORK.

Correct answers are +4 points each, wrong answers are -1 points each, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 5 will be added to your total grade only if it is positive.

TRUE FALSE **a**. Let (s_n) be a sequence of nonzero real numbers. Then, $\forall k \in \mathbb{R}$ ($\lim_{n\to\infty} s_n$ exists if and only if $\lim_{n\to\infty} ks_n$ exists).

TRUE FALSE **b**. Let (s_n) be a sequence of real numbers. If $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |s_n - L| < \varepsilon$, then the sequence (s_n) converges.

TRUE FALSE **c**. Let (s_n) and (t_n) be sequences of real numbers such that $\forall n \in \mathbb{N}$, $s_n \le t_n$. If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

TRUE FALSE **d**. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists n \in \mathbb{N}$ such that $((x > 0 \text{ and } y > 0) \Rightarrow x > \frac{y}{n})$.

TRUE FALSE **e**. Let (s_n) be a sequence of rational numbers, $\mathbb{Q}$. If $\forall \varepsilon > 0, \exists M \ni \forall m,n \in \mathbb{N} \ni (m,n \ge M \Rightarrow |s_n - s_m| < \varepsilon)$, then $\exists L \in \mathbb{Q}, \ni \forall \varepsilon > 0, \exists M \ni \forall n \in \mathbb{N}, (n \ge M \Rightarrow |s_n - L| < \varepsilon)$.