1. For each of the following statements,

(i) State whether it is true or false, and

(ii) Prove the statement if it is true, or give a counterexample disproving the statement with details if it is false.

If (i) is not stated, then (ii) will not earn credit.

a. Prove that if $\frac{x+4}{x} \le 3$, then $x \ge 2$ or $x < 0$, for $x \in \mathbb{R}$.

(i) True.

(ii) $(p \Rightarrow q \vee r)$ is equivalent to $(p \wedge \sim q \Rightarrow r)$

So it suffices to prove

$$\frac{x+4}{x} \le 3 \text{ and } x \geqslant 0 \implies x \geqslant 2.$$

Assume $\frac{x+4}{x} \le 3$. Since $\frac{x+4}{x}$ is defined, $x \neq 0$.

$$x > 0$$

$$x \cdot \frac{x+4}{\cancel{x}} \le 3 \cdot x$$

$$x+4 \le 3x$$

$$4 \le 2x$$

$$2 \le x. \qquad \#.$$

b. Assume that x and y real numbers. If x is an irrational number and y is an irrational number, then $x+y$ is an irrational number.

(i) False.

(ii) We proved in class that $\sqrt{2} \notin \mathbb{Q}$

Hence $-\sqrt{2} \notin \mathbb{Q}$.

$$\underbrace{\sqrt{2}}_{x} + \underbrace{(-\sqrt{2})}_{y} = \underbrace{0}_{x+y} \in \mathbb{Q}.$$

both irrational

is rational.

2. Let A,B, D and E be subsets of a universal set U.
For each of the following statements,
(i) State whether it is true or false, and
(ii) Prove the statement if it is true, or give a counterexample disproving the statement with details if it is false.
If (i) is not stated, then (ii) will not earn credit.

a. For every subset A and B a universal set U, one has $A \setminus (B \setminus A) = B \setminus (A \setminus B)$.

(i) False.

(ii) $A = \{0\}$, $B = \{1\}$

$B \setminus A = \{1\} \setminus \{0\} = \{1\}$

$A \setminus (B \setminus A) = \{0\} \setminus \{1\} = \{0\}.$

$A \setminus B = \{0\} \setminus \{1\} = \{0\}$

$B \setminus (A \setminus B) = \{1\} \setminus \{0\} = \{1\}$

$\neq$.

Actually any example with $A \neq B$ will disprove the statement (a), since $A \setminus (B \setminus A) = A$
$B \setminus (A \setminus B) = B$

b. Prove that if $E \subseteq D$, then $(U \setminus D) \subseteq (U \setminus E)$.

(i) True.

Assume $E \subseteq D$.

$\forall x \in U \; (x \in E \Rightarrow x \in D)$

The contrapositive is equivalent to the original.

$\forall x \in U \; (x \notin D \Rightarrow x \notin E)$

$x \in U \setminus D \iff (x \in U \text{ and } x \notin D)$

Hence $x \in U \setminus D \Rightarrow x \in U \setminus E$.

$U \setminus D \subseteq U \setminus E$.

3. You are asked to prove Theorems 2.3.16(a) and a part of Theorem 2.3.18(b). You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.3.16(a) or 2.3.18(b) or their consequences. Simply, saying "This follows Theorem 2.3.16(a) and 2.3.18(b)" will not earn any credit. You are expected to provide proofs.

DO ANY TWO OF (a), (b) and (c).

Let $f : A \to B$.

a. Prove that for every subset $C \subseteq A$, one has $C \subseteq f^{-1}[f(C)]$.

Proof If $C = \emptyset$, then $C = \emptyset \subseteq f^{-1}(f(C))$ holds.
If $C \neq \emptyset$, Let $a \in C$ be an arbitrary elt.

$$f(a) \in f(C) := D$$

$$f(a) \in D \iff a \in f^{-1}(D) = \{x \in A \mid f(x) \in D\}.$$

So $a \in f^{-1}(D) = f^{-1}(f(C))$.

$$C \subseteq f^{-1}(f(C)).$$

b. Prove that if f is surjective then for every subset $D \subseteq B$, one has $f[f^{-1}(D)] = D$. (You may assume Theorem 2.3.16(b): $f[f^{-1}(D)] \subseteq D$.)

Proof: Assume f is surjective (that is onto.)
Since we can assume $f(f^{-1}(D)) \subseteq D$, what remains to show is $D \subseteq f(f^{-1}(D))$. If $D = \emptyset$, then claim holds.
If $D \neq \emptyset$, Let $b \in D$ be an arbitrary element.
$b \in D \subseteq B$. f is onto B so $\exists$ $a \in A$ s.t. $f(a) = b$.
By defn of $f^{-1}(D) = \{x \in A \mid f(x) \in D\}$, $f(a) = b \in D$, one has $a \in f^{-1}(D)$
$b = f(a) \in f(f^{-1}(D)) \underset{\text{def}^n}{=} \{f(x) \mid x \in f^{-1}(D)\}$. Hence $\forall b (b \in D \Rightarrow b \in f(f^{-1}(D)))$
$D \subseteq f(f^{-1}(D))$.

c. Prove that if for every subset $C \subseteq A$ one has $C = f^{-1}[f(C)]$, then f is injective.

Proof Assume $\forall C \subseteq A$, $C = f^{-1}(f(C))$. $\circledast$
To show $\forall a, b \in A$, $f(a) = f(b) \Rightarrow a = b$. (Defn of injective)
Let $a, b \in A$ s.t. $f(a) = f(b) = k$. Take $C = \{a\}$.
$f(C) = f(\{a\}) = \{k\}$. Since $f(b) = k$ $b \in f^{-1}(\{k\})$.
$b \in f^{-1}(\{k\}) = f^{-1}(f(\{a\})) \underset{\circledast}{=} \{a\}$. $b \in \{a\} \Rightarrow b = a$.

C

4. a. State *The Well-Ordering Property of* $\mathbb{N}$ *(also known as The Axiom of Well-Ordering of* $\mathbb{N}$*).*

Every non-empty subset of $\mathbb{N}$ has a smallest element.

OR

$$\forall S \subseteq \mathbb{N} \ (S \neq \emptyset \Rightarrow \exists m \in S \ni \forall n \in S, \ m \leq n)$$

b. State *The Principle of Mathematical Induction.*

Let $p(n)$ be a statement for each $n \in \mathbb{N}$.

If (i) $p(1)$ is true, and (ii) $\forall k \in \mathbb{N} \ (p(k) \Rightarrow p(k+1))$, then

$p(n)$ is true for all $n \in \mathbb{N}$.

c. Prove *The Principle of Mathematical Induction,* by using *The Well-Ordering Property of* $\mathbb{N}$.

This is in the book & in the class notes.

Proof: by Contradiction

Assume: (i) $p(1)$ is true,

(ii) $\forall k \in \mathbb{N} \ (p(k) \Rightarrow p(k+1))$, and

(iii) Axiom of well ordering.

} Hypothesis.

Suppose $p(n)$ is not true for all $n \in \mathbb{N}$, we will try to find a contradiction. Suppose $\exists n_0 \in \mathbb{N}$ s.t. $p(n_0)$ is false. Define $S = \{n \in \mathbb{N} \mid p(n) \text{ is false.}\}$.

$S \neq \emptyset$ since $n_0 \in S$.

Axiom of Well-ordering $\Rightarrow \exists m \in S$ s.t. $\forall n \in S$, $m \leq n$.

$1 \notin S$ since $p(1)$ is true.

$m \neq 1$.

$m-1 \in \mathbb{N}$ since $m \in \mathbb{N}$, and $m > 1$.

$p(m-1)$ is a statement.

$m-1 \notin S$ since m is a minimum of S.

$p(m-1)$ is true,

$p(m-1) \Rightarrow p(m)$ by (ii)

Contradiction. $\Bigg[$ $p(m)$ is true by the last two lines

$p(m)$ is false since $m \in S$.

This proves that $S = \emptyset$ and $p(n)$ is true for all $n \in \mathbb{N}$.

5. TRUE OR FALSE

CIRCLE YOUR ANSWERS. NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.

Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: **Read very carefully.**

TRUE FALSE

a. The negation of the statement $\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, (x+y<5 \Rightarrow \exists z \in \mathbb{R} \ni (xy^2<1 \text{ and } x>z))$ is equivalent to $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \ni (x+y<5 \text{ and } (\forall z \in \mathbb{R}, (xy^2 \geq 1 \text{ or } x \leq z)))$.

actual negation of the first statement $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x+y<5 \text{ and } (\forall z \in \mathbb{R}\ (xy^2 \geq 1 \text{ or } x \leq z$

TRUE FALSE **b.** The set of rational numbers $\mathbb{Q}$ and the set of integers $\mathbb{Z}$ are equinumerous, even though $\mathbb{Q}$ contains all of $\mathbb{Z}$ and many other numbers.

$\exists$ bijection $\mathbb{N} \xrightarrow{f} \mathbb{Z}$

$\exists$ bijection $\mathbb{N} \xrightarrow{g} \mathbb{Q}$.

$\Rightarrow$ $\mathbb{Z} \underset{1-1,\text{onto}}{\xrightarrow{f^{-1}}} \mathbb{N} \underset{1-1,\text{onto}}{\xrightarrow{g}} \mathbb{Q}$ is a bijection.

TRUE FALSE **c.** Only one of the following two statements (i) and (ii) is true:

i. $\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, \exists z \in \mathbb{R} \ni x(y^2+z^2)=0.$

ii. $\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R}, \exists z \in \mathbb{R} \ni x(y^2+z^2) \neq 0.$

Both are true.

(i) $\exists\ x=0\ \forall y\ \exists z$ any number $0\cdot(y^2+z^2)=0$

(ii) $\exists\ x=1\ \forall y\ \exists z=1 \quad 1\cdot(\underbrace{y^2+1}_{>0}) \neq 0$

TRUE FALSE **d.** For every set A, in order to prove "$\forall n \in A, p(n)$" is false, it suffices to show that "$\exists n \notin A \ni \sim p(n)$" is true.

Correct: It suffices to show $\exists n \in A \sim p(n)$.

What happens outside the set A, does not say anything about what happens in A.

TRUE FALSE **e.** For every two real numbers x and y, one has $|x|-|y| \leq |x-y|$.

$|x|-|y| \leq ||x|-|y|| \leq |x-y|$

$a \leq |a|$.

reverse Δ-ineq.