3.1 "Log n" means "ln n" in Rudin.

(a) Use Thm 3.27

(*) $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log(\log n))^p}$ converges

$\Longleftrightarrow \sum_{k=2}^{\infty} \frac{\cancel{2^k}}{\cancel{2^k}(\log 2^k)(\log(\log 2^k))^p}$

$= \sum_{k=2}^{\infty} \frac{1}{(k \log 2)(\log k + \underbrace{\log\log 2}_{<0})^p}$

$= \frac{1}{\log 2} \sum_{k=2}^{\infty} \frac{1}{k(\log k + c)^p}$ converges

$c<0$

$\frac{1}{k(\log k)^p} \leq \frac{1}{k(\log k + c)^p} \leq \frac{1}{(\frac{1}{2})^p k (\log k)^p}$

Thm 3.29 $\sum \frac{1}{k(\log k)^p}$ converges $\Longleftrightarrow$ $p>1$

Hence (*) converges $\Longleftrightarrow$ $p>1$.
by using comparison test & Thm 3.27.

3.1 (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n (x-1)^n}{\log n}.$

$$\sqrt[n]{\left|\frac{(-2)^n}{\log n}\right|} = \frac{2}{(\log n)^{\frac{1}{n}}} \longrightarrow 2 \quad \text{since}$$

$$1 \leq \log n \leq n \quad \text{for } n \geq 3$$

$$1 \leq \sqrt[n]{\log n} \leq \sqrt[n]{n} \longrightarrow 1 \quad \text{Thm 3.20}$$

$$\sqrt[n]{\log n} \longrightarrow 1.$$

Radius of convergence $R = \frac{1}{2}$ Thm 3.39.

$|x-1| > \frac{1}{2}$ divergent

$|x-1| < \frac{1}{2} \implies$ convergent

$\frac{1}{2} < x < \frac{3}{2}$ $\quad (\frac{1}{2}, \frac{3}{2}) \subseteq$ Domain of convergence $\subseteq [\frac{1}{2}, \frac{3}{2}]$

Check end pts.

$x = \frac{1}{2}$. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n (\frac{1}{2}-1)^n}{\log n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$

diverges since $n > \log n$

$\frac{1}{n} < \frac{1}{\log n}$

$\therefore$ compare to $\sum \frac{1}{n}$ diverges

$x = \frac{3}{2}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n (\frac{3}{2}-1)^n}{\log n} = \sum \frac{(-1)^n}{\log n}.$ converges

by Alternating series test. Domain of Convergence $= (\frac{1}{2}, \frac{3}{2}]$.

3.2

$\bar{A} = \{\frac{(-1)^n}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

As in the proof of Thm 2.12 use a diagonal exhaustion.

$$\begin{array}{cccccccccc}
 & 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} & \frac{1}{6} & -\frac{1}{7} & \cdots \\
① & 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} & \frac{1}{6} & -\frac{1}{7} & \\
② & 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} & \frac{1}{6} & -\frac{1}{7} & \\
③ & 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} & \frac{1}{6} & -\frac{1}{7} & \\
④ & & & & & & & & &
\end{array}$$

$$\underbrace{0}_{①}, \underbrace{-1, 0}_{②}, \underbrace{\frac{1}{2}, -1, 0}_{③}, \underbrace{-\frac{1}{3}, \frac{1}{2}, -1, 0}_{④}, \underbrace{\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}, -1, 0 \ldots}_{⑤}$$

Every element of $\bar{A}$ appears in this sequence infinitely many times, by taking such a subsequence of constant entries we see that $\bar{A} \subseteq L$.

On the other hand $\forall\, p \notin \bar{A}$ which is closed $\exists\, \varepsilon > 0$ $N_\varepsilon(p) \cap \bar{A} = \emptyset$. ($\mathbb{R} - \bar{A}$ is open) Hence no sequence using elements of $\bar{A}$ can converge to p.

$L \subseteq \bar{A}$.

(b) $\limsup s_n = \max \bar{A} = \frac{1}{2}$ $\qquad$ $\limsup s_n^2 = 1$

$\liminf s_n = \min \bar{A} = -1$ $\qquad$ $\liminf s_n^2 = 0$

3.3 $f: X \to X$, $d(f(x), f(y)) \le r\, d(x,y)$
for $0 \le r < 1$, $r \in \mathbb{R}$ fixed

$x_{n+1} = f(x_n)$, x_0 arbitrary

(a) Let $d(x_0, x_1) = A$

$d(x_1, x_2) = d(f(x_0), f(x_1)) \le r\, d(x_0, x_1) = rA$ (*)

Claim $\forall n \in \mathbb{N}$ $d(x_n, x_{n+1}) \le r^n A$.

Proof by Induction

$n=1$ done above. (*)

$d(x_n, x_{n+1}) \le r^n A \Rightarrow d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1}))$
$\le r\, d(x_n, x_{n+1}) \le r \cdot r^n A = r^{n+1} A$ #

If $m > n$,

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\le r^n A + r^{n+1} A + \cdots + r^{m-1} A$$
$$= A r^n (1 + r + r^2 + \cdots + r^{m-1-n})$$
$$\le A r^n \frac{1}{1-r}$$ since $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $0 \le r < 1$.

$\forall \varepsilon > 0$ given, choose N s.t. $r^N < \frac{\varepsilon(1-r)}{A}$ since $r^n \to 0$ for $0 \le r < 1$.

Hence $\forall \varepsilon > 0\ \exists N\ \forall_{\substack{n,m > N \\ m > n}}$

$$d(x_n, x_m) \le A \frac{r^n}{1-r} \le A \frac{r^N}{1-r} < \varepsilon$$

$\Rightarrow \{x_n\}$ is Cauchy in (X, d)

(b) (X, d) Complete $\xRightarrow[\text{(def}^n\text{)}]{} \{x_n\}$ is convergent

3.3 (c)

Given: $x_0 \neq x_0'$, $x_{n+1}' = f(x_n')$
$x_{n+1} = f(x_n)$ $\forall n \geq 0$

Suppose $p = \lim_n x_n \neq \lim_n x_n' = q$.

Let $\varepsilon = d(p,q)/3 > 0$

$\exists N_1 \ \forall n \geq N_1 \ d(x_n, p) < \varepsilon$
$\exists N_2 \ \forall n \geq N_2 \ d(x_n', q) < \varepsilon$

$\forall n \geq N = \max(N_1, N_2)$
$d(p,q) \leq d(p, x_n) + d(x_n, x_n') + d(x_n', q)$

$3\varepsilon < \varepsilon + d(x_n, x_n') + \varepsilon$
$0 < \varepsilon < d(x_n, x_n') \ \forall n \geq N$

$$0 < \varepsilon < d(x_{n+m}, x_{n+m}') \leq r^m \underbrace{d(x_n, x_n')}_{\text{fixed} > 0 \text{ since } \varepsilon > 0} \xrightarrow{\text{as } m \to \infty} 0$$

Contradiction.
Hence $p = q = \lim_n x_n = \lim_n x_n'$.

3.4 Exc. #23 p 82

$\{p_n\}, \{q_n\}$ Cauchy $\Rightarrow d(p_n, q_n)$ converges.
in (X, d)

$\forall n$ $d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$

$d(p_n, q_n) - d(p_m, q_m) \le d(p_n, p_m) + d(q_m, q_n)$

interchange p_n, q_n with p_m, q_m

$d(p_m, q_m) - d(p_n, q_n) \le d(p_m, p_n) + d(q_n, q_m)$

(*) $|d(p_m, q_m) - d(p_n, q_n)| \le d(p_m, p_n) + d(q_n, q_m)$

$\left.\begin{array}{lll} \{p_n\} \text{ Cauchy} & \forall \varepsilon > 0 \ \exists N_1 \ \forall m, n \ge N_1 & d(p_n, p_m) < \frac{\varepsilon}{2} \\ \{q_n\} \text{ Cauchy} & \forall \varepsilon > 0 \ \exists N_2 \ \forall m, n \ge N_2 & d(q_n, q_m) < \frac{\varepsilon}{2} \end{array}\right\} (**)$

Let $N = \max(N_1, N_2)$

$\forall n, m \ge N$

$|d(p_m, q_m) - d(p_n, q_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ by (*) (**)

$\Rightarrow \{d(p_m, q_m)\}_{m=1}^{\infty}$ is Cauchy

d takes values in $\mathbb{R}$, which is complete.

$\Rightarrow d(p_m, q_m) \longrightarrow d_0 \in \mathbb{R}$.

Caution $\lim p_n$, $\lim q_n$ in X may not exist.

3.5 Exercise 3.7

$$\left.\begin{array}{l}\sum a_n \text{ converges} \\ a_n \geq 0\end{array}\right\} \Longrightarrow \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \text{ converges.}$$

Method I $\quad 0 \leq \left(\sqrt{a_n} - \frac{1}{n}\right)^2 = a_n - 2\frac{\sqrt{a_n}}{n} + \frac{1}{n^2}$

$$0 \leq \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right).$$

$\sum a_n$, $\sum \frac{1}{n^2}$ convergent $\Longrightarrow \sum \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$ converges

$\Longrightarrow \sum \frac{\sqrt{a_n}}{n}$ converges by Comparison test.

METHOD II $\quad S_m = \sum_{n=1}^{m} \frac{\sqrt{a_n}}{n} = (\sqrt{a_1}, \sqrt{a_2}, .. \sqrt{a_m}) \cdot \left(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m}\right)$

Cauchy Schwartz $\leq \|(\sqrt{a_1}, \sqrt{a_2}, .. \sqrt{a_m})\| \cdot \|(1, \frac{1}{2}, \cdots \frac{1}{m})\|$

$$= \underbrace{\left(\sum_{n=1}^{m} a_n\right)^{1/2} \left(\sum_{n=1}^{m} \frac{1}{n^2}\right)^{1/2}} \leq \underbrace{\left(\sum_{n=1}^{\infty} a_n\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}}_{A < \infty.}$$

If you claim a version in $\mathbb{R}^\infty$, you must prove it.

(both converges as $m \to \infty$
$a_n \geq 0$, $\frac{1}{n^2} \geq 0$)

$S_m = \sum_{n=1}^{m} \frac{\sqrt{a_n}}{n} \uparrow$ since $\frac{\sqrt{a_n}}{n} \geq 0$

$S_m \leq A < \infty.$

$\Longrightarrow \lim S_m$ exists. $\quad \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty.$

4.1 $d(x,y) = |x-y|$ on $\mathbb{R} \times \mathbb{Q}$.

a) $\left.\begin{array}{l}\mathbb{R} \text{ connected} \\ f \text{ continuous}\end{array}\right\} \Rightarrow f(\mathbb{R})$ is connected.

Suppose $\exists\, a, b \in f(\mathbb{R})$, $a < b$
By density of irrationals in $\mathbb{R}$: $\exists c$, $a < c < b$
and $c \notin \mathbb{Q}$.

$a \in A = (-\infty, c) \cap \mathbb{Q} \cap f(\mathbb{R}) \neq \emptyset$
$b \in B = (c, \infty) \cap \mathbb{Q} \cap f(\mathbb{R}) \neq \emptyset$.
$A \cap B = \emptyset$., $A \cup B = f(\mathbb{R})$
$\bar{A} \subseteq (-\infty, c]$, $\bar{B} \subseteq [c, \infty)$ in $\mathbb{Q}$.
$\bar{A} \cap \bar{B} = \emptyset$. since $c \notin \mathbb{Q}$.

$\Rightarrow f(\mathbb{R})$ is not connected. Contradiction.
$\Rightarrow f(\mathbb{R})$ cannot have more than one pt in it.
$\Rightarrow f(\mathbb{R})$ is constant.

Not required in the test:
$\left.\begin{array}{l} g: \mathbb{Q} \to \mathbb{R} \text{ unif continuous} \\ \{x_n\} \text{ Cauchy}\end{array}\right\} \Rightarrow \{g(x_n)\}$ Cauchy

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \mathbb{Q}\ \ |x-y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$
$\{x_n\}$ Cauchy Given δ above $\exists N\ \forall n, m \geq N\ \ |x_n - x_m| < \delta$.
$\Rightarrow |g(x_n) - g(x_m)| < \varepsilon$.

Hence
$\forall \varepsilon > 0\ \exists N\ \forall n, m \geq N\ \ |g(x_n) - g(x_m)| < \varepsilon$
$\{g(x_n)\}$ Cauchy.

(b) $g : \mathbb{Q} \to \mathbb{R}$ uniformly continuous

Prove $\exists\, G : \mathbb{R} \to \mathbb{R}$ continuous s.t.

$$G(x) = g(x) \ \forall x \in \mathbb{Q}.$$

Defining G:

If $x \in \mathbb{Q}$, then take $G(x) = g(x)$.

If $x \notin \mathbb{Q}$, take any $x_n \to x$, $x_n \in \mathbb{Q}$ by density of $\mathbb{Q}$ in $\mathbb{R}$.

$\{x_n\}$ Cauchy in $\mathbb{R}$

$\{x_n\}$ Cauchy in $\mathbb{Q}$.

$\{g(x_n)\}$ Cauchy in $\mathbb{R}$

$\lim_{n \to \infty} g(x_n) = L$ for some $L \in \mathbb{R}$.

Define $g(x) = L$.

G is Well-defined: Suppose $\exists\, x_n \to x$, $x_n \in \mathbb{Q}$

$\exists\, y_n \to x$ $y_n \in \mathbb{Q}$.

$$\left.\begin{array}{l} g(x_n) \to L_1 \\ g(y_n) \to L_2 \end{array}\right\} \text{ and } L_1 \neq L_2.$$

Let $\varepsilon = |L_1 - L_2| > 0$

(*) $\exists\, \delta > 0$ $\forall a, b \in \mathbb{Q}$, $|a - b| < \delta \Rightarrow |g(a) - g(b)| < \frac{\varepsilon}{3}$

$\exists N_1$ $\forall n \geq N_1$ $|x_n - x| < \frac{\delta}{2}$ and $|g(x_n) - L_1| < \frac{\varepsilon}{3}$

$\exists N_2$ $\forall n \geq N_2$ $|y_n - x| < \frac{\delta}{2}$ and $|g(y_n) - L_2| < \frac{\varepsilon}{3}$

Let $N = \max(N_1, N_2)$

$\forall n \geq N$ $\quad |x_n - y_n| \leq |x_n - x| + |x - y_n| < \delta$

$$\Rightarrow |g(x_n) - g(y_n)| < \frac{\varepsilon}{3} \text{ by (*)}$$

$$\varepsilon = |L_1 - L_2| \le |L_1 - g(x_n)| + |g(x_n) - g(y_n)| + |g(y_n) - L_2|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Contradiction.

Hence G is well-defined.

(Uniform) Continuity of $G(x)$.

Let $\varepsilon > 0$ be given.

g unif cont $\Rightarrow \exists\, \delta' > 0 \;\; \forall a, b \in \mathbb{Q}$

(*) $\quad |a-b| < \delta' \Rightarrow |g(a) - g(b)| < \frac{\varepsilon}{3}$.

For Continuity of G, take $\delta = \frac{\delta'}{3} > 0$.

$\forall x, y \in \mathbb{R}, \quad |x-y| < \delta = \frac{\delta'}{3}$

$\left(\begin{array}{l} \text{If } x \in \mathbb{Q} \;\; x_n = x \\ \text{If } y \in \mathbb{Q} \;\; y_n = y \end{array}\right) \begin{cases} \exists\, x_n \to x, \; x_n \in \mathbb{Q}, \text{ so that } g(x_n) \to G(x) \\ \exists\, y_n \to y, \; y_n\, \mathbb{Q} \text{ so that } g(y_n) \to G(y) \end{cases}$

① $\exists N_1 \; \forall n \ge N_1 \quad |g(x_n) - G(x)| < \frac{\varepsilon}{3}$ and $|x_n - x| < \frac{\delta'}{3}$

② $\exists N_2 \; \forall n \ge N_2 \quad |g(y_n) - G(y)| < \frac{\varepsilon}{3}$. and $|y_n - y| < \frac{\delta'}{3}$

$$|x_n - y_n| \le |x_n - x| + |x - y| + |y - y_n| < \delta'$$

③ $\quad |g(x_n) - g(y_n)| < \varepsilon/3$ by (*)

$|G(x) - G(y)| \le$

$\le |G(x) - g(x_n)| + |g(x_n) - g(y_n)| + |g(y_n) - G(y)|$

$\le \underbrace{\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}}_{①-②-③} = \varepsilon \qquad \#$

4.2 Exercise 4.10 Prove Thm 4.19

Complete details: write a whole proof, Including the justifications of the hints provided by the book.

We want to prove

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in X$$
$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

Suppose not

$$\exists \varepsilon_0 \ \forall \delta = \tfrac{1}{n} \ \exists x_n, y_n \in X \qquad (**)$$
$$d_X(x_n, y_n) < \tfrac{1}{n} \text{ and } d_Y(f(x_n), f(y_n)) \geq \varepsilon_0$$

Since X is compact

$\exists x_{n_k}$ subsequence of x_n s.t. $x_{n_k} \to p_0$.

Caution(!) Look at y_{n_k} in X, $\exists$ a subsequence $y_{n_{k_\ell}}$ of y_{n_k} s.t. $y_{n_{k_\ell}} \to q_0$

$$x_{n_k} \to p_0 \Rightarrow \text{Also } x_{n_{k_\ell}} \to p_0$$

Set $\quad p_\ell = x_{n_{k_\ell}} \to p_0$

$$q_\ell = y_{n_{k_\ell}} \to q_0$$

$$d(p_\ell, q_\ell) < \frac{1}{n_{k_\ell}} \leq \frac{1}{\ell}$$

since $\quad n_{k_\ell} \geq k_\ell \geq \ell$

Claim $p_0 = q_0$: $\forall \varepsilon > 0$

$$d_X(p_0, q_0) \le d_X(p_0, p_\ell) + d_X(p_\ell, q_\ell) + d_X(q_\ell, q_0)$$

$\exists N \ \forall \ell \ge N$ $\quad d(p_\ell, p_0) < \frac{\varepsilon}{3}$, and
$d(q_\ell, q_0) < \varepsilon/3$, and
$d(p_\ell, q_\ell) < 1/\ell \le \varepsilon/3$

$\forall \varepsilon > 0 \quad d_X(p_0, q_0) \le \varepsilon.$
$\Rightarrow \quad d_X(p_0, q_0) = 0$
$\Rightarrow \quad p_0 = q_0$

$(**) \Rightarrow \quad \varepsilon_0 \le d(f(p_\ell), f(q_\ell))$
$\le d(f(p_\ell), f(p_0)) + d(f(p_0), f(q_\ell))$ (p_ℓ, q_ℓ are subsequences of x_n, y_n)

Continuity of f at p_0: For ε_0 we have
$\exists N_1' \ \forall \ell \ge N_1'$ $\left.\begin{array}{l} d(f(p_\ell), f(p_0)) < \varepsilon_0/3 \\ \lim_{\ell \to \infty} f(p_\ell) = f(p_0) \end{array}\right\}$

$\exists N_2' \ \forall \ell \ge N_2' \quad d(f(q_\ell), f(\underset{p_0 = q_0}{p_0})) < \frac{\varepsilon_0}{3}.$

$$\varepsilon_0 \le d(f(p_\ell), f(q_\ell)) \le d(f(p_\ell), f(p_0)) + d(f(p_0), f(q_\ell))$$
$$< \frac{2\varepsilon_0}{3}$$

$0 < \varepsilon_0 < \frac{2\varepsilon_0}{3}$ Contradiction.

#

4.3 Exercise 4.5

Prove $\left.\begin{array}{l} E \text{ closed} \subseteq \mathbb{R} \\ f: E \to \mathbb{R} \text{ continuous.} \end{array}\right\} \Rightarrow \exists g: \mathbb{R} \to \mathbb{R}$ continuous,

s.t. $g(x) = f(x)$ $\forall x \in E$.

Soln $\mathbb{R} - E$ is open.

E is the union of at most countable collection of disjoint open intervals (Discussed in Exc. 2.29, HW)

$$\mathbb{R} - E = \underbrace{\left(\bigcup_{i=1}^{\infty} (a_i, b_i) \right) \cup (-\infty, \inf E) \cup (\sup E, \infty)}_{\text{some of which are empty: } (a,a) = \emptyset}$$

If $E = \emptyset$, then g can be any continuous function

WLOG: $E \neq \emptyset$.

$\forall i$, $a_i, b_i \in E$ unless, $a_i = b_i$ and $\sup E$, $\inf E$ are in E unless $\pm\infty$.

If $\inf E > -\infty$, define g on $(-\infty, \inf E)$ to be $f(\inf E)$.

If $\sup E < \infty$, define g on $(\sup E, \infty)$ to be $f(\sup E)$.

$\forall\, x \in (a_i, b_i)$, $\quad a_i, b_i \in E$, $\quad a_i < b_i$

define

$$g(x) = f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i)$$

(linear function)

Observe that

$g(x)$ is between $f(a_i)$ and $f(b_i)$.

g is continuous on each (a_i, b_i) if $\neq \emptyset$.

- g is continuous on $E^c = \mathbb{R} - E$ since $\forall x \in E^c$ open, $\quad \exists\, \delta_0 > 0 \;\; N_{\delta_0}(x) \subseteq E^c$

g on $N_{\delta_0}(x)$ is linear. Continuity of g on $N_{\delta_0}(x)$ is the same as its continuity on $\mathbb{R}$, ($N_{\delta_0}(x)$ open)

- g is continuous on $\operatorname{int} E$ since

$g = f$ on $\operatorname{int} E$ and

$\forall x \in \operatorname{int} E$, $\exists\, \delta_0 > 0 \;\; N_{\delta_0}(x) \subseteq E$.

The continuity of f and g on $N_{\delta_0}(x)$ are the same as long as

$\forall \varepsilon > 0 \;\; \exists\, \delta > 0$, $\;\delta_0 \geq \delta > 0$

$|x - y| < \delta \implies |g(x) - g(y)| = |f(x) - f(y)| < \varepsilon$.

- Remains to prove <u>continuity of g on $\operatorname{bd}(E)$</u>.

Let $x \in \operatorname{bd}(E)$ be fixed, let $\varepsilon > 0$ be given

$f(x) = g(x)$ since $x \in \operatorname{bd}(E) \subseteq E$

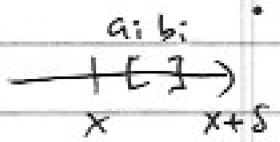
and E is closed.

By continuity of f on E at $x \in E$:

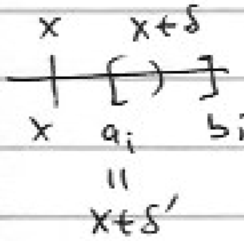
$\exists \delta > 0, \forall y \in E \quad |x-y| < \delta \implies$

$$|f(x) - f(y)| = |g(x) - g(y)| < \varepsilon$$

<u>Case</u> where $y \notin E$ and $y \in N_\delta(x)$



- If $\phi \neq [a_i, b_i] \subseteq N_\delta(x)$, then
  $f(a_i) = g(a_i)$ and $f(b_i) = g(b_i) \in N_\varepsilon(f(x))$
  g linear on $[a_i, b_i]$,
  $g([a_i, b_i])$ is between $g(a_i)$ & $g(b_i)$
  $g([a_i, b_i]) \subseteq N_\varepsilon(f(x))$ (interval)
  $\forall y \in [a_i, b_i]$, $g(y) \in N_\varepsilon(f(x))$:
  $$|g(x) - g(y)| < \varepsilon.$$



- $[a_i, b_i] \cap N_\delta(x) \neq \phi$ but $[a_i, b_i] \notin N_\delta(x)$

namely $x \leq a_i < x + \delta \leq b_i$ OR
$a_j \leq x - \delta < b_j \leq x$

If $x \neq a_i$, choose $\delta' = \min(\delta, a_i - x) > 0$.
If $x \neq b_j$, choose $\delta' = \min(\delta, x - b_j) > 0$.
$$|x-y| < \delta' \implies |g(x) - g(y)| < \varepsilon.$$

If $x = a_i$ or $x = b_j$, then use the continuity of linear functions to take
$$\delta' = \min(\delta, \varepsilon/(|m_i|+1)), \quad m_i = \frac{f(b_i) - f(a_i)}{b_i - a_i}$$
$|x-y| < \delta' \implies |g(x) - g(y)| < \varepsilon$.

- $f(x) = \frac{1}{x} : (0,1) \to \mathbb{R}$ doesn't extend to $\mathbb{R}$.

- $f = (f_1, f_2, \dots f_k) : E \to \mathbb{R}^k$. Construct extensions g_i of each $f_i : E \to \mathbb{R}$.

4.4 Exercise 4.6

Graph $= G = \{(x, f(x)) \mid x \in E\} \subseteq X \times Y$

There are many ways to put a metric on $X \times Y$

- $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$
- $d_0((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

Obs. $\lim (x_n, y_n) = (\lim x_n, \lim y_n)$ w.r.t. d, d_0.

It is fine if this is not addressed, and proved for $f: E \subseteq \mathbb{R} \to \mathbb{R}$, with standard $\mathbb{R}^2$.

(i) To prove E compact, f continuous $\Rightarrow G$ compact

Define $g(x) = (x, f(x)) : E \to X \times Y$.

If $x \in E'$: $\lim\limits_{n \to \infty} g(x_n) = \lim\limits_{n \to \infty} (x_n, f(x_n))$

$= (\lim x_n, \lim f(x_n)) = (x, f(x)) = g(x)$

g is continuous on E, since it is automatically continuous on $E - E'$.

E compact, g continuous $\Rightarrow G = g(E)$ is compact.

Thm 4.14.

(ii) To prove: E compact, G compact $\Rightarrow f$ continuous.

$$g(x) = (x, f(x)) : E \to G = \{(x, f(x)) \mid x \in E\}$$

g is defined, onto G by its definition,
1-1 $((x_1, f(x_1)) = (x_2, f(x_2)) \Rightarrow x_1 = x_2)$

Hence $h = g^{-1}$ exists: $h(x, f(x)) = x$.

$h_0(x,y) = x$ is continuous on $X \times Y \to X$
since $(x_n, y_n) \to (x_0, y_0) \Rightarrow x_n \to x_0$,
by either d or d_0 metric

h is continuous, $h : G \to E$,
G compact, h is 1-1 and onto.
Thm 4.17 $\Rightarrow h^{-1} = g$ is continuous.
$g(x) = (x, f(x))$ continuous
$\Rightarrow f(x)$ continuous

since:

$$g(x) = (x, f(x)) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} (x_n, f(x_n))$$

$$= \left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} f(x_n)\right) = (x, f(x))$$

$=$

f is continuous at $x \in E'$.
Continuity is automatic if $x \notin E'$.

4.5 $\rho_E(x) = \inf_{z\in E} d(x,z)$ in (X, d)

$\rho_E : X \longrightarrow \mathbb{R}$.

(a) $x \in \bar{E} \iff \forall r>0\ \exists z \in N_r(x) \cap E$

$\iff \forall r>0\ \exists z\in E$ s.t. $d(x,z)<r$

$\iff \forall r>0\ \inf_{z\in E} d(x,z) < r$

$\iff \inf_{z\in E} d(x,z) = 0 \iff \rho_E(x) = 0$.

(b) $\forall x, y \in X, \forall z \in E$

$\rho_E(x) \le d(x,z) \le d(x,y) + d(y,z)$

independent of z $\longrightarrow$ $\rho_E(x) - d(x,y) \le d(y,z)$

$\rho_E(x) - d(x,y) \le \inf_{z\in E} d(y,z) = \rho_E(y)$

$\rho_E(x) - \rho_E(y) \le d(x,y)$.

$\rho_E(y) - \rho_E(x) \le d(y,x)$ by symmetry

$|\rho_E(x) - \rho_E(y)| \le d(x,y)$.

$\forall \varepsilon>0\ \exists \delta = \varepsilon > 0\ \forall x,y \in X$

$d(x,y) < \delta \implies |\rho_E(x) - \rho_E(y)| < \delta = \varepsilon$.

#21. (c) K compact, F closed $F\cap K = \phi = \bar{F} \cap \bar{K}$

$\rho_F : X \longrightarrow \mathbb{R}$, K compact.

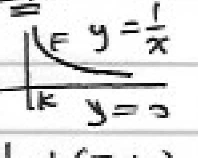
$\rho_F > 0$ on K. $\rho_F(K) \subseteq (0, \infty)$

K compact $\longrightarrow$ $\inf K = \min K \in K$.

$\implies \exists \delta>0,\ \rho_F(K) \subseteq [\delta, \infty)$.

$\forall p \in K\ \forall q \in F\quad d(p,q) \ge \rho_F(p) \ge \delta > \frac{\delta}{2} > 0$.

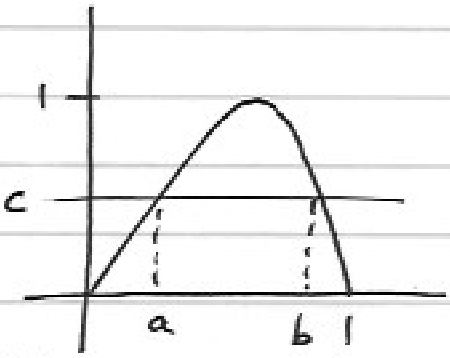
Ex



dist(F, K) = 0.

TRUE/ False

(a) False



$a \neq b$
$f(a) = f(b) = c.$

$$x_n = \begin{cases} a & \text{if } n \text{ odd} \\ b & \text{if } n \text{ even.} \end{cases}$$

$x_n \not\to$ anything
$f(x_n) = c \to c.$

(b) True (Thm.)

(c) False $f: \mathbb{R} \to [0,1]$
$f(x) \equiv 1$
f is continuous, $f^{-1}(\{1\}) = \mathbb{R}$
not compact

False

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ converges by Alternating series test

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$