MATH 2850
MIDTERM 2
April 14, 2017

NAME.____ SOLUTION

SIGNATURE.________________________

Do all **5** problems, **20 points each**.

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example: $\sqrt{2}$, $\ln 2$, e^3 or $\sin\frac{\pi}{8}$), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

DO NOT WRITE BELOW:

1.______

2.______

3.______

4.______

5.______

Total.______

Problem 1.
Let $f(x,y) = e^x(x^2+y^2)$
a. Calculate ∇f and Hf.

$$\nabla f = \left(e^x(x^2+y^2+2x),\ 2ye^x\right)$$

$$H_f = \begin{bmatrix} e^x(x^2+y^2+2x+2x+2) & 2ye^x \\ 2ye^x & 2e^x \end{bmatrix}$$

b. Calculate the second order Taylor polynomial of f at $(x,y) = (0,1)$.

$$f(0,1) = 1 \qquad H_f(0,1) = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\nabla f(0,1) = (1,2)$$

$$P_2 = 1 + 1\cdot(x-0) + 2(y-1) + \frac{1}{2}\left(3(x-0)^2 + 4(x-0)(y-1) + 2(y-1)^2\right)$$

c. Identify all critical points of f and determine the nature of each critical point (local maxima, local minima, saddle points).

$$\nabla f = 0 \iff \begin{cases} e^x(x^2+y^2+2x) = 0 \\ 2ye^x = 0 \end{cases} \iff \begin{matrix} x^2+y^2+2x = 0 \\ 2y = 0 \end{matrix}$$

since $e^x > 0\ \forall x$

$$y = 0 \implies x^2+2x = 0 \implies x = 0, -2 \qquad \left.\begin{matrix}(0,0)\\(-2,0)\end{matrix}\right\} \text{c.p.}$$

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{matrix}\Delta_1 = 2 > 0 \\ \Delta_2 = 4 > 0\end{matrix} \qquad (0,0) \text{ local min.}$$

$$H_f(-2,0) = \begin{bmatrix} e^{-2}(4-8+2) & 0 \\ 0 & 2e^{-2} \end{bmatrix} = \begin{bmatrix} e^{-2}(-2) & 0 \\ 0 & 2e^{-2} \end{bmatrix}$$

$$\Delta_1 = -2e^{-2} < 0$$
$$\Delta_2 = -4e^{-4} < 0 \implies \text{Saddle at } (-2,0).$$

Problem 2. Find the absolute maximum and minimum values of $f(x,y) = x^2 + 2y^2 - 2x$ on the closed disk $\{(x,y) \mid x^2 + y^2 \leq 4\}$. State where the maximum and minimum values are attained.

$\nabla f = (2x-2, 4y)$ $\quad \nabla f = 0 \iff \begin{array}{l} 2x-2=0 \\ 4y = 0 \end{array}$

$\iff (1,0)$ interior cp.

$g = x^2+y^2 = 4$

$\nabla g = (2x, 2y)$

L.M.

$\nabla f = \lambda \nabla g$ $\quad \begin{cases} 2x-2 = 2\lambda x & ① \\ 4y = 2\lambda y & ② \\ x^2+y^2 = 4. & ③ \end{cases}$

② $\Rightarrow$ $4y - 2\lambda y = 0 = 2y(2-\lambda)$

$\Rightarrow$ $y=0$ or $\lambda = 2$.

Case 1 $\quad y = 0$

③ $\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$.

Case 2 $\quad \lambda = 2$

$2x-2 = 2\lambda x = 4x$

$-2 = 2x$

$-1 = x$

③ $\Rightarrow$ $y = \pm\sqrt{3}$.

		x^2+2y^2-2x	
int.	$(1,0)$	$1-2=-1$	min
	$(2,0)$	$4-4=0$	
	$(-2,0)$	$4+4=8$	
	$(-1,\sqrt{3})$	$1+6+2=9$	max
	$(-1,-\sqrt{3})$	$1+6+2=9$	max

Since $\{(x,y) \mid x^2+y^2 \leq 4\}$ is closed & bounded, f must have a max and a min value on it.

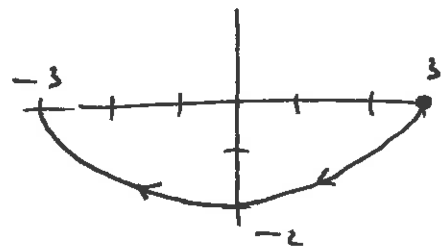
Max value 9 at $(-1, \pm\sqrt{3})$

Min value -1 at $(1,0)$.

Problem 3. Let $\mathbf{x}(t) = 3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$, for $0 \le t \le \pi$.

a. (i) Sketch the image of the path $\mathbf{x}(t)$, using arrows to indicate its direction.

$$\vec{x}(t) = (3\cos t, -2\sin t)$$
$$\vec{x}(0) = (3, 0)$$
$$\vec{x}(\pi/2) = (0, -2)$$
$$\vec{x}(\pi) = (-3, 0)$$



ii. Find a parametric equation for the tangent line to $\mathbf{x}(t)$ when $t = \frac{\pi}{6}$.

$$\vec{x}\left(\tfrac{\pi}{6}\right) = \left(3\cos\tfrac{\pi}{6}, -2\sin\tfrac{\pi}{6}\right) = \left(\tfrac{3\sqrt{3}}{2}, \underbrace{-2\left(\tfrac{1}{2}\right)}_{-1}\right)$$

$$\vec{x}'(t) = (-3\sin t, -2\cos t)$$
$$\vec{x}'\left(\tfrac{\pi}{6}\right) = \left(-\tfrac{3}{2}, -\tfrac{2\sqrt{3}}{2}\right)$$
$$\ell(s) = \left(\tfrac{3\sqrt{3}}{2}, -1\right) + s\left(-\tfrac{3}{2}, -\sqrt{3}\right)$$

b. **(i)** Why does the equation $xz^4 + y^3z + x^5y = 3$ implicitly define z as a function of x and y near the point $(x,y,z) = (1,1,1) = \mathbf{p}$?

(ii) If $g(x,y) = z$ is the solution function near $\mathbf{p}$, what is Dg at $\mathbf{p}$? Compute the derivative matrix Dg near $\mathbf{p}$ in terms of x, y, and z.

Let $F = xz^4 + y^3z + x^5y : \mathbb{R}^3 \to \mathbb{R}$.

$$DF = [\,z^4 + 5x^4y \quad 3y^2z + x^5 \quad 4xz^3 + y^3\,]$$

$$DF(1,1,1) = [\,6 \quad 4 \quad 5\,]$$ want to solve for z

$\det[5] = 5 \neq 0$, hence the $\exists\ g(x,y) = z$ solving z near $(1,1,1)$

$$Dg(1,1,1) = -[5]^{-1}[\,6 \quad 4\,] = -\left[\tfrac{1}{5}\right][\,6 \quad 4\,] = \left[-\tfrac{6}{5} \quad -\tfrac{4}{5}\right].$$

$$Dg(x,y,z) = -[4xz^3 + y^3]^{-1}[\,z^4 + 5x^4y \quad 3y^2z + x^5\,]$$
$$= \left[-\frac{z^4 + 5x^4y}{4xz^3 + y^3} \quad -\frac{3y^2z + x^5}{4xz^3 + y^3}\right].$$

Problem 4.

Consider the integral $\int_0^4 \int_0^{\sqrt{y}} 2x\sin(y^2)\,dx\,dy$.

a. Evaluate this integral directly.

$$\int_0^4 \int_0^{\sqrt{y}} 2x \sin y^2 \,dx\,dy = \int_0^4 x^2 \sin y^2 \Big|_{x=0}^{x=\sqrt{y}} dxy$$

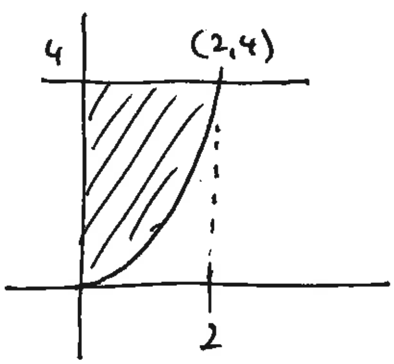
$$= \int_0^4 \left(\left(\sqrt{y}\right)^2 \sin y^2 - 0\right) dy = \int_0^4 y \sin y^2 \, dy$$

Let $u = y^2$
$du = 2y\,dy$

$$= \int_0^{16} \frac{1}{2} du \sin u \,\cancel{du} = -\frac{1}{2}\cos u \Big|_0^{16} = -\frac{1}{2}(\cos 16 - 1)$$

$$= \frac{1}{2} - \frac{\cos 16}{2}.$$

b. Sketch the domain of integration.



$0 \le y \le 4$
$0 \le x \le \sqrt{y}$

$x = \sqrt{y} \iff x^2 = y$

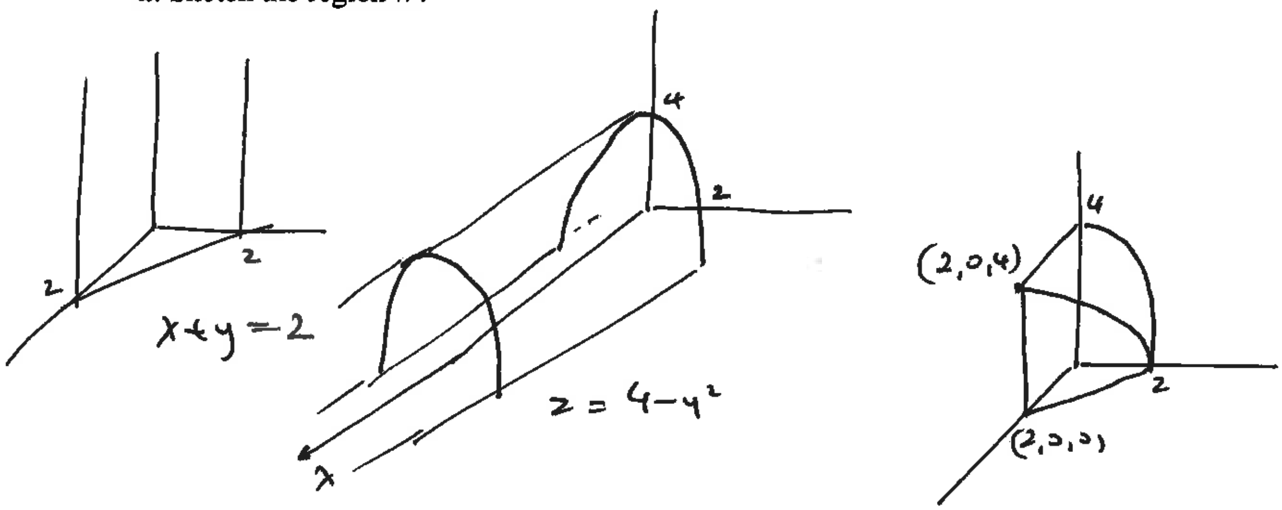
c. Reverse the order of integration of the integral above.

$$\int_0^2 \int_{x^2}^4 2x \sin(y^2)\, dy\, dx.$$

$0 \le x \le 2$
$x^2 \le y \le 4$

Problem 5. Let W be the region in the first octant of $\mathbf{R}^3$ $(x,y,z \geq 0)$, bounded by the cylinder $z = 4 - y^2$, and the planes $x + y = 2$, $x = 0$, $y = 0$, and $z = 0$.

a. Sketch the region W.



b. Calculate $\iiint_W y^2\, dV$.

$$\int_0^2 \int_0^{2-x} \int_0^{4-y^2} y^2 \, dz\, dy\, dx = \int_0^2 \int_0^{2-x} y^2 z \Big|_{z=0}^{z=4-y^2} dy\, dx$$

$$= \int_0^2 \int_0^{2-x} y^2(4-y^2)\, dy\, dx = \int_0^2 \int_0^{2-x} 4y^2 - y^4 \, dy\, dx$$

$$= \int_0^2 \frac{4y^3}{3} - \frac{y^5}{5} \Big|_0^{2-x} dx = \int_0^2 \frac{4}{3}(2-x)^3 - \frac{1}{5}(2-x)^5 \, dx$$

Let $u = 2-x$
$du = -dx$

$$= -\int_2^0 \frac{4}{3} u^3 - \frac{1}{5} u^5 = \frac{1}{3} u^4 - \frac{1}{30} u^6 \Big|_0^2 = \frac{16}{3} - \frac{64}{30} = \frac{16}{5}.$$

(PTO) for other orders:

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} y^2 \, dz \, dx \, dy$$

$$= \int_0^2 \int_0^{2-y} y^2(4-y^2) \, dx \, dy$$

$$= \int_0^2 y^2\underbrace{(4-y^2)(2-y)}_{8-4y-2y^2+y^3} \, dy$$

$$= \int_0^2 8y^2 - 4y^3 - 2y^4 + y^5 \, dy$$

$$= \frac{8}{3}y^3 - y^4 - \frac{2}{5}y^5 + \frac{y^6}{6} \Big|_0^2 = \frac{64}{3} - 16 - \frac{64}{5} + \frac{\cancel{64}^{32}}{\cancel{6}_3}$$

$$= \frac{96}{3} - 16 - \frac{64}{5} = 16 - \frac{64}{5}$$

$$= \frac{80-64}{5} = \frac{16}{5}$$

OR

$$\int_0^2 \int_0^{4-y^2} \int_0^{2-y} y^2 \, dx \, dz \, dy$$

$$= \int_0^2 \int_0^{4-y^2} y^2(2-y) \, dz \, dy$$

$$= \int_0^2 y^2(2-y)(4-y^2) \, dy$$

$$= \int_0^2 8y^2 - 4y^3 - 2y^4 + y^5 \, dy \overset{\text{as above}}{=} \cdots = \frac{16}{5}$$