Sept 8, 2017

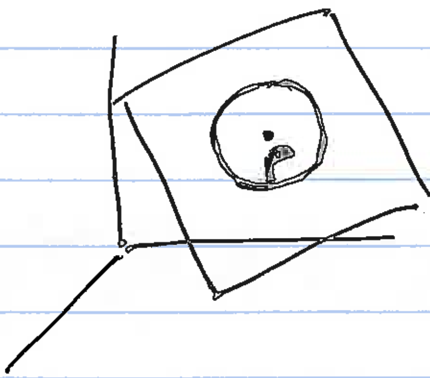
①

Theorem: Let $\beta \in C^3$, $|\beta'(s)| \equiv 1$, $\beta \subseteq \mathbb{R}^3$.

β is a part of a circle of radius R

$\iff$ $\tau_\beta \equiv 0$ and $\kappa_\beta \equiv \frac{1}{R}$

Proof ($\Rightarrow$:) Assume: β is a circle of radius R.



ⓐ $\|\beta(s) - p\| = R$.

center

ⓑ $\beta \subseteq$ plane, $\tau \equiv 0$

$$f(s) = \|\beta(s) - p\|^2 = R^2 \quad \text{constant}$$

① $0 = f'(s) = 2 \cdot T \cdot (\beta - p)$

② $0 = f''(s) = 2\left[\kappa N \cdot (\beta - p) + 1\right]$

③ $0 = f'''(s) = 2\left[\kappa' N - \kappa^2 T + \kappa \underset{0 \;\; \tau \equiv 0}{\tau} B\right] \cdot (\beta - p)$

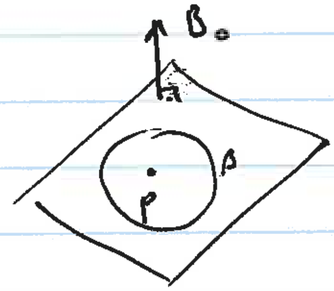
① $T \cdot (\beta - p) = 0 \quad T \perp \beta - p$.

$\kappa \neq 0 \Leftarrow$ ② $N \cdot (\beta - p) = -\frac{1}{\kappa} \neq 0$

③ $\kappa' \overbrace{N \cdot (\beta - p)}^{\neq 0} = 0$

$\Rightarrow \kappa' = 0$, κ is constant

WTS $k = \frac{1}{R}$



$\beta(s) \subseteq$ plane $\perp$ $B_0 = B(s)$

$\beta - p \perp B = B_0$

$\beta - p \perp T \Leftarrow T \cdot (\beta - p) = 0$

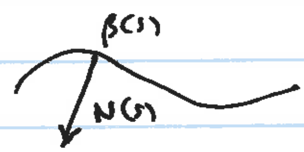
$\beta - p \parallel N$

$$\frac{1}{k} = \left|-\frac{1}{k}\right| \overset{(2)}{=} \left|(\beta - p)\cdot N\right| = \underbrace{|\beta - p|}_{R} \cdot \underbrace{|N|}_{1}\underbrace{|\cos\theta|}_{1} = R$$

$k = \frac{1}{R}$ ✓

Proof ($\Leftarrow$:) $\tau_\beta \equiv 0$, $k_\beta \equiv \frac{1}{R} \overset{WTS}{\Longrightarrow}$ circle of radius R

$\tau_\beta \equiv 0 \Rightarrow$ plane curve.



Let $\gamma(s) = \beta(s) + \frac{1}{k_\beta} N(s)$

$\gamma(s) = \beta(s) + R N(s)$

$\gamma'(s) = \beta'(s) + R \cdot (\underbrace{-kT + \cancel{\tau}^{0} B}_{N'})$

$\gamma'(s) = T - \underbrace{Rk}_{1} T = 0$

$\gamma(s)$ constant $= p_0$.

$\gamma(s) = p_0 = \beta(s) + R N(s)$

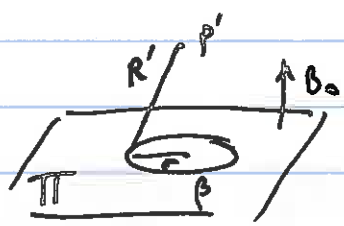
$$\beta(s) - p_0 = R \underbrace{N(s)}_{\text{unit}}$$

$$\|\beta(s) - p_0\| = R.$$

$\beta(s)$ Lies on a sphere of radius R centered at p_0.

$\tau \equiv 0 \implies \beta(s)$ lies on a plane.

$\beta(s)$ is a circle. How do we know p_0 is on the same plane?



$$\gamma(s) = \beta(s) + \rho N(s) = p_0$$

$\beta(s) \subseteq$ plane $\Pi \perp B_0$

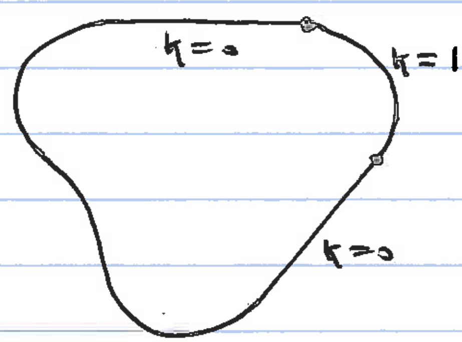
$N \parallel$ plane Π $\quad (N \perp B_0)$

$\implies p_0 \in$ plane Π

$\implies$ radius of circle $r = R$.

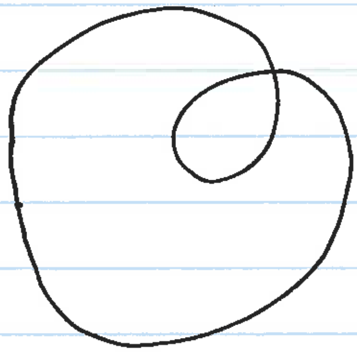
Examples:

closed
but
not C^1 closed



C^1-closed
but not
C^2 closed

<u>Example</u> Let $\beta(s)$ be a C^2 closed curve, $|\beta'(s)|=1$



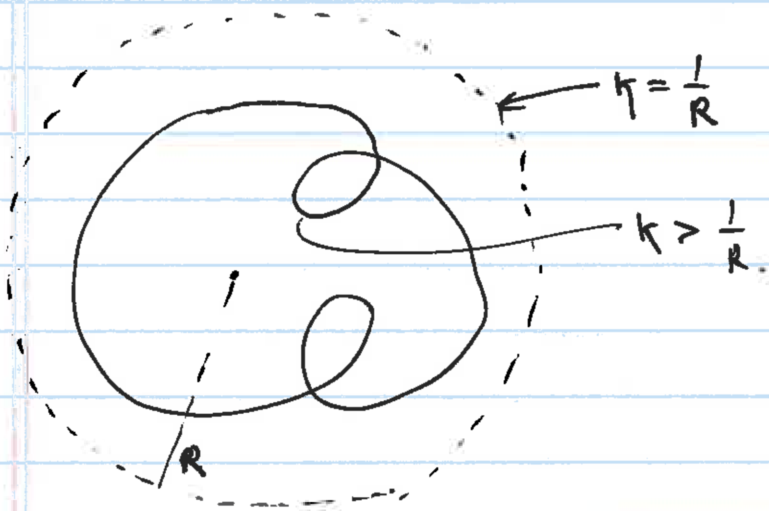
$\beta : [a,b] \longrightarrow \mathbb{R}^n \quad n = 2,3$

$\beta(a) = \beta(b)$ (closed)
$\beta'(a) = \beta'(b)$ (C^1-closed)
$\beta''(a) = \beta''(b)$ (C^2-closed)

Let $\beta(s) \subseteq B_R(0) = \{ \vec{x} \in \mathbb{R}^n \mid \|x\| < R \}$

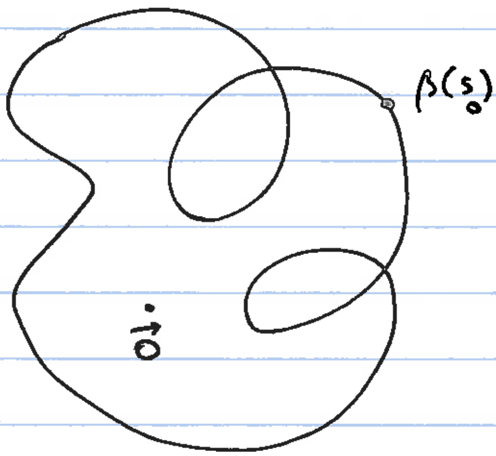
$\Longrightarrow \quad \exists \, s_0 \quad \kappa_\beta(s_0) > \frac{1}{R}.$

<u>In $\mathbb{R}^2$</u>



<u>Proof:</u> let $f(s) = \|\beta(s)\|^2 = \text{dist}^2(\beta(s), 0)$

Lemma $\Rightarrow$
$$\begin{cases} f'(s) = 2T \cdot \beta \\ f''(s) = 2[T' \cdot \beta + 1] \end{cases}$$
$$= 2[\kappa N \cdot \beta + 1].$$



Let $\beta(s_0)$ be the furthest pt from the origin.

(closed curve $\Rightarrow$ image is compact.)

$\Rightarrow \exists$ a furthest pt from $\vec{0}$.

$f(s_0)$ max of $f(s)$

$$f'(s_0) = 0$$

$$f''(s_0) \leq 0$$

$$[\kappa N \cdot \beta + 1](s_0) \leq 0$$

$$1 \leq -\kappa(s_0) \cdot N(s_0) \cdot \beta(s_0)$$

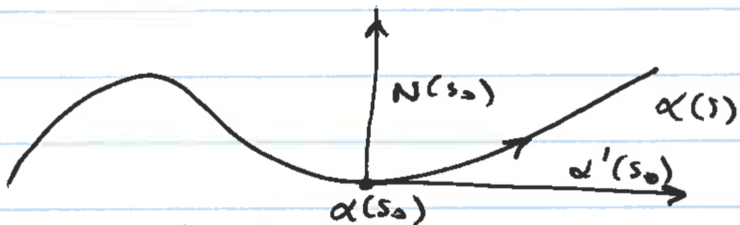
$$1 \leq |\kappa(s_0)| \, |N(s_0) \cdot \beta(s_0)|$$

$$1 \leq |\kappa(s_0)| \, \underbrace{|N(s_0)|}_{1} \, \underbrace{|\beta(s_0)|}_{< R}$$

$$1 \lneqq \kappa(s_0) R$$

$$\frac{1}{R} \lneqq \kappa(s_0) \checkmark$$

Defn Let $\alpha(s)$ be a regular curve



Eqⁿ of tangent line to α at $\alpha(s_0)$

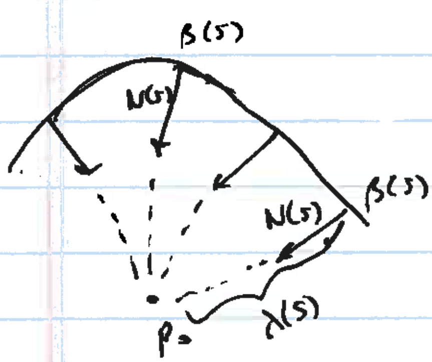
$$\ell(t) = \alpha(s_0) + t \cdot \alpha'(s_0)$$

If $\kappa(s_0) > 0$, $\alpha \in C^2$,

Eqⁿ of the normal line:

$$\ell(t) = \alpha(s_0) + t \cdot N(s_0)$$

Exc Let $\beta(s)$ be a C^3 curve, $\kappa(s) > 0$, $|\beta'(s)| \equiv 1$. If all normal lines pass through a fixed pt p_0 $(p_0 \notin \beta)$ then $\beta(s)$ is a circle



Proof:

$$\beta(s) + \lambda(s) N(s) = p_0$$

$$\beta' + \lambda' N + \lambda N' = 0.$$

$$T + \lambda' N + \lambda(-\kappa T + \tau B) = 0.$$

$$T(1-\kappa\lambda) + \lambda' N + \lambda\tau B = 0.$$

T, N, B lin independent

① $1 - k\lambda = 0$

② $\lambda' = 0$ $\qquad p_0 \notin \beta$

③ $\lambda\tau = 0 \quad (\lambda \neq 0) \Rightarrow \tau = 0.$

② $\Rightarrow$ λ constant

① $1 = \kappa\lambda$, $\underline{\kappa \text{ constant}}$, $\kappa \neq 0$

By Thm.

circle.

Actually: $\|\beta(s) - p_0\| = \|\lambda\|$ constant

$\Rightarrow$ β is a part of the circle with center p_0 and radius λ.

<u>Remark:</u> We do not need to assume $p_0 \notin \beta$, in the statement.

Since the above argument shows that All parts of β away from p_0 stays on a circle of fixed positive radius. By continuity β can never take value p_0.