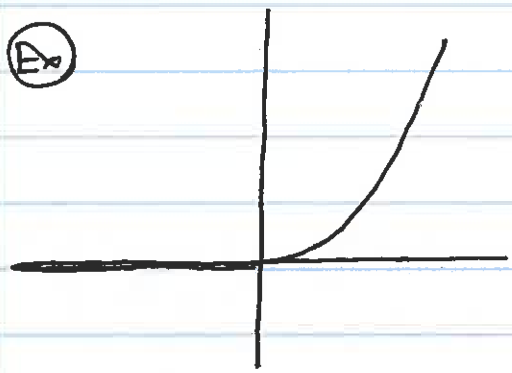
1.3

Thm: Let $\beta : I \longrightarrow \mathbb{R}^3$ be C^3 and $|\beta'(s)| = 1$

1) $\kappa(s) \equiv 0 \iff \beta(s) = p_0 + v_0 s$, a line. $(|v_0| = 1)$

2) If $\kappa(s) > 0$ for all s, then $\tau(s) \equiv 0 \iff \beta(s)$ is a plane curve

Ex

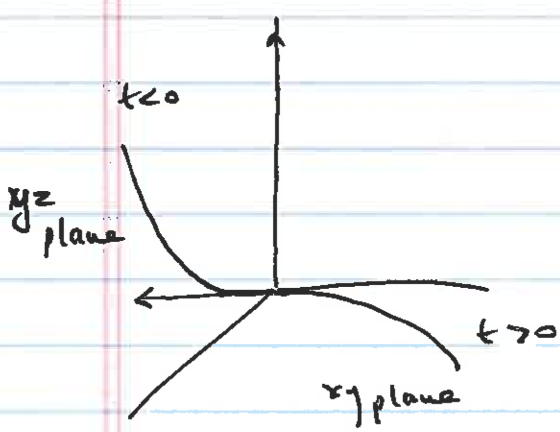


$$f(t) = \begin{cases} 0 & t \le 0 \\ e^{-\frac{1}{t^2}} & t > 0 \end{cases}$$

$f \in C^\infty$

$f^{(k)}(0) = 0 \ \forall k \in \mathbb{N}$.

f is not analytic i.e. $\exists$ no $R > 0$ s.t. Taylor series of f equals to f for $|t| < R$.



Let
$$\alpha(t) = \begin{cases} (0, t, e^{-\frac{1}{t^2}}) & t < 0 \\ (0, 0, 0) & t = 0 \\ (e^{-\frac{1}{t^2}}, t, 0) & t > 0 \end{cases}$$

$\alpha(t),\ t > 0 \subseteq xy$ plane

$\alpha(t)\ t < 0 \subseteq yz$ plane

$\boxed{\Longrightarrow \tau \equiv 0 \text{ for } t \ne 0.}$

But $\{\alpha(t) \mid t \in \mathbb{R}\}$ does not lie in any plane

Why? $\kappa(0) = 0$, undefined $\tau(0)$ (Of course this curve needs to be parametrized wrt arclength.

Proof:

1) $0 \equiv \kappa(s) \equiv |T'(s)| \equiv |\beta''(s)|$

$\beta''(s) = 0 \iff \beta'(s) = v_0$ for some $v_0 \in \mathbb{R}^3$

$\iff \beta(s) = v_0 s + p_0$ for some $p_0, v_0 \in \mathbb{R}^3$

2) Assume $\kappa(s) > 0 \ \forall s$.
WTS $\tau \equiv 0 \implies \beta$ is a plane curve.

$\tau \equiv 0$

$B' = -\tau N \equiv 0$

$B = B_0$ constant ($\neq 0$ since $|B| = 1$)

not same B's, use b

Recall Calc III Plane $ax + by + cz = D$

$(a, b, c) \cdot (x, y, z) = D$

$u_0 \cdot \vec{x} = c_0$

Let $f(s) = \beta(s) \cdot B_0$

$f'(s) = \beta'(s) \cdot B_0 = T(s) \cdot B(s) = 0$

$f(s)$ constant

$\beta(s) \cdot B_0 = c_0$

$\implies \beta(s) \subseteq$ plane whose equation

$\vec{x} \cdot B_0 = c_0$.

$\kappa(s) > 0 \ \forall s$ (3)

WTS:

2) β is a plane curve $\Longrightarrow$ $\tau \equiv 0$

$\beta \subseteq$ plane $\vec{u}_0 \cdot \vec{x} = c_0$.

$$\vec{\beta}(s) \cdot \vec{u}_0 = c_0 \qquad u_0 \text{ constant } \neq \vec{0}.$$

$$\left.\begin{aligned} T(s)\cdot\vec{u}_0 &= \beta'(s)\cdot\vec{u}_0 = 0 \\ 0 &= (T(s)\cdot\vec{u}_0)' = \kappa(s) N(s)\cdot\vec{u}_0 \\ &\quad 0<\kappa(s)\ \forall s \\ &\Longrightarrow N(s)\cdot\vec{u}_0 = 0 \end{aligned}\right] \longrightarrow \begin{aligned} &u_0 \perp T(s) \\ &u_0 \perp N(s) \\ &u_0 \parallel T\times N = B \end{aligned}$$

$$B(s) = \pm \frac{u_0}{|u_0|} \quad \forall s \quad \text{due to continuity of } B(s)$$

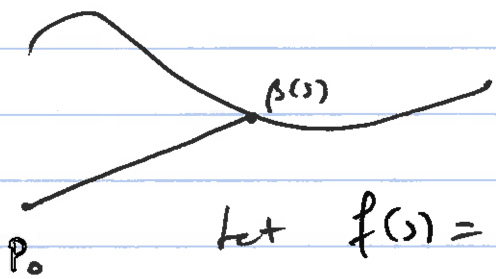
$$\Uparrow$$

$$\beta \in C^3.$$

$B(s)$ constant $= B_0$.

$$\left.\begin{aligned} -\tau N = B'(s) = 0 \\ N \neq 0 \end{aligned}\right\} \Longrightarrow \tau \equiv 0.$$

Lemma: Let $\beta(s) \in \mathbb{E}^3$, $|\beta'(s)| \equiv 1$.



Let $f(s) = \operatorname{dist}^2(p_0, \beta(s))$
$= |\beta(s) - p_0|^2$

Then $f'(s) = 2T(s) \cdot (\beta(s) - p_0)$

$f''(s) = 2\left[\kappa(s) N(s) \cdot (\beta(s) - p_0) + 1\right]$

$$f'''(s) = 2\left[\kappa'(s) N(s) - \kappa^2(s) T(s) + \kappa(s) \tau(s) B(s)\right] \cdot (\beta(s) - p_0)$$

Proof. $f = (\beta - p_0) \cdot (\beta - p_0)$

$$f' = 2\, T \cdot (\beta - p_0)$$

$$f'' = 2 \cdot \left[T' \cdot (\beta - p_0) + T \cdot (\beta - p_0)'\right]$$

$$= 2\left[\kappa N \cdot (\beta - p_0) + \underbrace{T \cdot T}_{1}\right]$$

$$= 2\left[\kappa N \cdot (\beta - p_0) + 1\right].$$

$$f''' = 2\Big[\kappa' \cdot N(\beta - p_0) + \kappa \cdot N'(\beta - p_0) + \underbrace{\kappa N \cdot \underbrace{(\beta - p_0)'}_{T}}_{0}$$

$$= 2\Big[\kappa' N \cdot (\beta - p_0) + \kappa(-\kappa T + \tau B) \cdot (\beta - p_0) + 0\Big]$$

$$= 2\Big[\kappa' N - \kappa^2 T + \kappa \tau B\Big] \cdot (\beta - p_0).$$