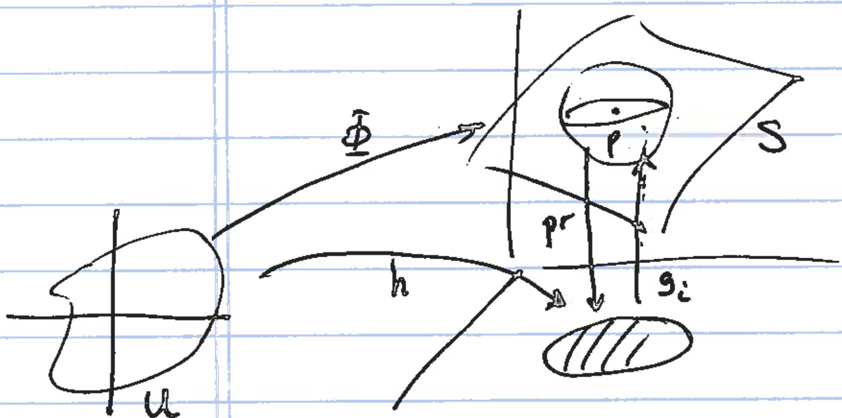
2.1½ Continue

P $\to$ E $\to$ I

Prop $\forall p \in S$ (regular surface), For a regular parametrization about p,



$\Phi : U \to S$

$\exists\, V$ open set $\subseteq \mathbb{R}^3$, $p \in V$

$\exists$ function in the form

$z = g_1(x,y)$ or

$y = g_2(x,z)$ or

$x = g_3(y,z)$ s.t.

the explicit graph of g_i represents S locally in V, about p.

Let $\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$

Main idea

$$\Phi_u \times \Phi_v = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$$= \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \underbrace{\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}}_{\frac{\partial(x,y)}{\partial(u,v)}} \right) \neq \vec{0}$$

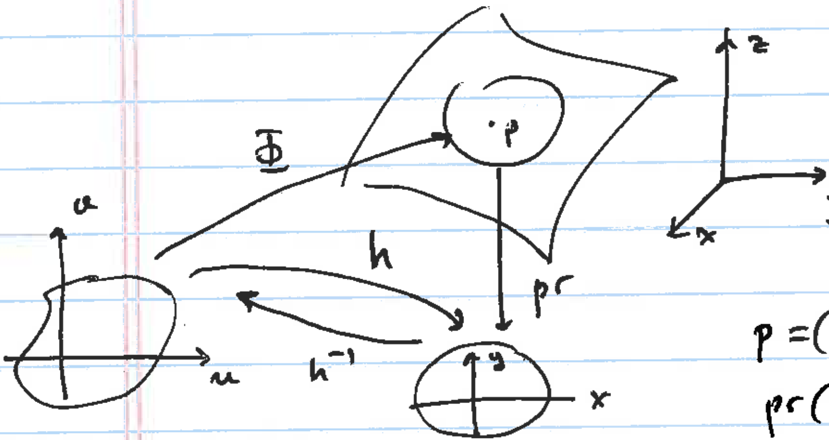
At least one of the components is not 0

Say $\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \neq 0$

Let $pr(x,y,z) = (x,y)$

Let $h = pr \circ \Phi$ $\qquad h(u,v) = (x(u,v),\ y(u,v))$

$$Dh = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \to \det \neq 0 \text{ at } \Phi^{-1}(p) = (u_0, v_0)$$



$p = (a, b, c) = \Phi(u_0, v_0)$

$pr(p) = (a, b)$

By Inv F. Thm $\exists$ local inverse of h around (u_0, v_0) : h^{-1}

Let $\quad f(x,y) = \Phi \circ h^{-1}$

$$pr \circ f(x,y) = \underbrace{pr \circ \Phi}_{h} \circ h^{-1}_{(x,y)} = h \circ h^{-1}_{(x,y)} = Id(x,y) = (x,y)$$

So: $f(x,y) = (x, y, g(x,y))$

(with some more work) S is locally the graph of $z = g(x,y)$.

$$h^{-1}, \phi, pr \in C^\infty \implies f \in C^\infty$$
$$g \in C^\infty.$$

This is main idea of

$$(P) \Longrightarrow (E) \xrightarrow[\text{easy}]{} (I)$$

Ex $\psi(u,v) = (u^3 + v^2,\ uv,\ e^{uv})$

$$\psi_u = (3u^2,\ v,\ ve^{uv})$$
$$\psi_v = (2v,\ u,\ ue^{uv})$$

$$D\psi = \begin{bmatrix} 3u^2 & 2v \\ v & u \\ ve^{uv} & ue^{uv} \end{bmatrix}$$

We proceed as in previous page

$$f = \psi \circ h^{-1} \qquad h(u,v) = (pr \circ \psi)(u,v) = (u^3+v^2,\ uv)$$

$$Df = D\psi \cdot Dh^{-1} = D\psi \cdot (Dh)^{-1}$$

$$Df = \begin{bmatrix} 3u^2 & 2v \\ v & u \\ ve^{uv} & ue^{uv} \end{bmatrix} \begin{bmatrix} 3u^2 & 2v \\ v & u \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{bmatrix}$$

$$\begin{bmatrix} ve^{uv} & ue^{uv} \end{bmatrix} \begin{bmatrix} 3u^2 & 2v \\ v & u \end{bmatrix}^{-1} = \begin{bmatrix} g_x & g_y \end{bmatrix}$$

$$\begin{bmatrix} g_x & g_y \end{bmatrix} = \begin{bmatrix} ve^{uv} & ue^{uv} \end{bmatrix} \begin{bmatrix} u & -2v \\ -v & 3u^2 \end{bmatrix} \frac{1}{3u^3 - 2v^2}$$

$$= \frac{1}{3u^3 - 2v^2} \begin{bmatrix} uve^{uv} - uve^{uv} & -2v^2e^{uv} + 3u^3e^{uv} \end{bmatrix}$$

g is indep of x.

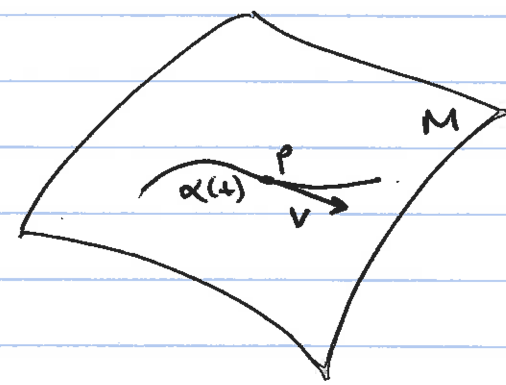
$$[g_x \quad g_y] = [0 \quad e^{uv}].$$

(*)(*) $$g(x,y) = e^{y} \quad \text{since } y = uv.$$

$$S \subseteq \{(x,y,z) \mid z = e^{y}\}.$$

Remark: One can always find $[g_x \; g_y]$, but (*)(*) is not easy to obtain most of the time. This example was set up in a way that finding $g(x,y)$ explicitly was possible.

3.2



Defn Let M be a regular surface, $M \subseteq \mathbb{R}^3$, $p \in M$. A vector $v \in \mathbb{R}^3$ is called a tangent vector to M at p if $\exists\ \alpha : (-\varepsilon, \varepsilon) \longrightarrow M$, α diff'ble s.t.

$$\alpha(0) = p$$
$$\alpha'(0) = v.$$

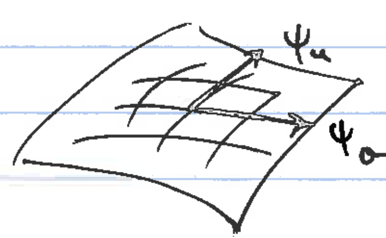
We denote v_p to indicate the pt of tangency, p.

Defn $T_pM = \{ v \in \mathbb{R}^3 \mid v \text{ is tangent to } M \text{ at } p \}$.

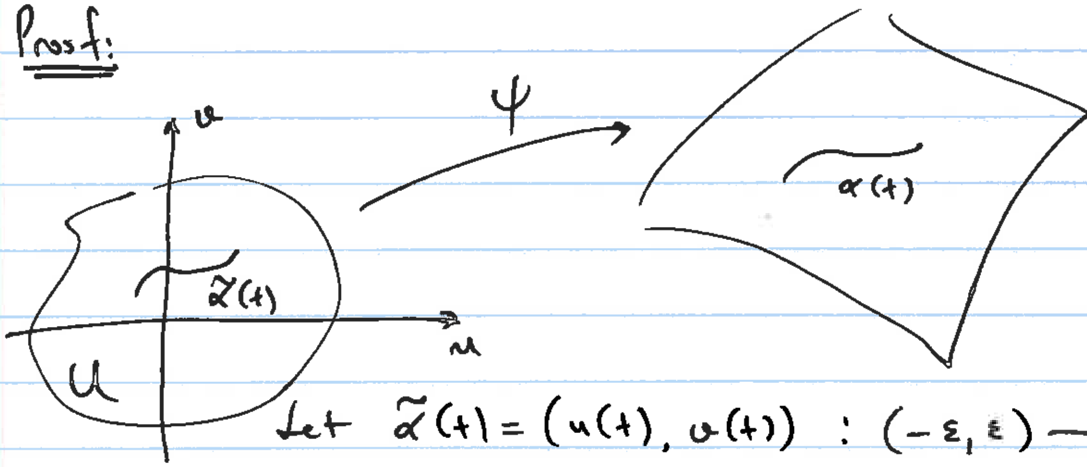
Called the tangent plane to M at p.

Prop $v \in T_pM \iff v = (c_1 \psi_u + c_2 \psi_v)(u_0, v_0)$ for all parametrizations ψ of M s.t. $\psi(u_0, v_0) = p$.

Corollary: if $\left.\begin{matrix} \psi_u \times \psi_v \neq 0 \\ \psi(u_0, v_0) = p \end{matrix}\right\}$ then $\{\psi_u(u_0, v_0), \psi_v(u_0, v_0)\}$ is a basis for T_pM.



Proof:



Let $\tilde{\alpha}(t) = (u(t), v(t)) : (-\varepsilon, \varepsilon) \longrightarrow U \subseteq \mathbb{R}^2$

$$\alpha(t) = (\psi \circ \tilde{\alpha})(t)$$

$$\psi = \psi(u, v)$$

Chain Rule: $\alpha'(t) = \psi_u \cdot u' + \psi_v \cdot v'$

$$(*) \quad \alpha'(0) = \psi_u(u_0, v_0) \cdot u'(0) + \psi_v(u_0, v_0) \cdot v'(0)$$

$$\alpha(0) = (u_0, v_0) = p$$

WTS: $T_pM \overset{?}{=} \{ c_1 \psi_u + c_2 \psi_v \mid c_1, c_2 \in \mathbb{R} \}$

Given $c_1, c_2 \in \mathbb{R}$, take

$$\tilde{\alpha}(t) = (u_0 + c_1 t, v_0 + c_2 t)$$

$$\tilde{\alpha}'(0) = (c_1, c_2)$$

Let $\alpha(t) = (\psi \circ \tilde{\alpha})(t)$

by $(*)$ $\alpha'(0) = c_1 \psi_u + c_2 \psi_v$

$\Rightarrow T_pM \supseteq \{ (c_1 \psi_u + c_2 \psi_v)(u_0, v_0) \mid c_1, c_2 \in \mathbb{R} \}$

where $\psi(u_0, v_0) = p$.

WTS "$\subseteq$" next

Let $v_p \in T_pM$, then $\exists\, \alpha(t) : (-\varepsilon, \varepsilon) \longrightarrow M$

s.t. $\alpha(0) = p$

$\alpha'(0) = v$.

Let $\tilde{\alpha} = \varphi^{-1} \circ \alpha$ so that

$\alpha = \varphi \circ \tilde{\alpha}$

$\tilde{\alpha}'(0) = (c_1, c_2)$ for some $c_1, c_2 \in \mathbb{R}$.

By ⊛ $\alpha'(0) = (c_1 \varphi_u + c_2 \varphi_v)(u_0, v_0) = v$

where $\varphi(u_0, v_0) = p$.

$\Rightarrow \quad T_pM \subseteq \{ (c_1 \varphi_u + c_2 \varphi_v)(u_0, v_0) \mid c_1, c_2 \in \mathbb{R} \}$.

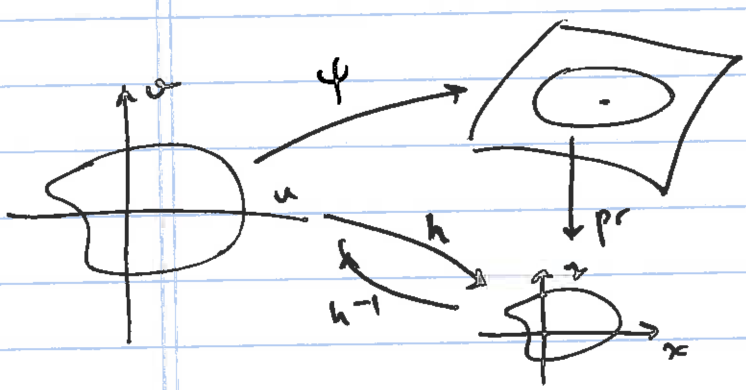
<u>Caution</u> There are some hidden facts that make this work:

Q1. Why does φ^{-1} exist?

A1 By defn of Surfaces.

Q2 why does $\varphi^{-1} \circ \alpha$ diff'ble?

A.2 Go to page 2 of today's lecture:



$pr \circ \varphi = h$.

$\exists\, h^{-1}$ by Inv F. Thm locally

$h^{-1} \circ pr = \varphi^{-1}$

$\underbrace{h^{-1} \circ pr \circ \alpha}_{\text{diff'ble}} = \varphi^{-1} \circ \alpha$