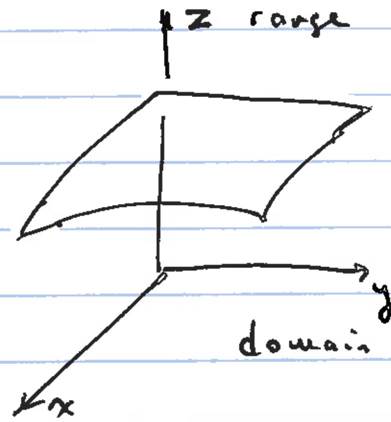
2.1½

There are 3 basic different types of graphs:
We will look at them in $\mathbb{R}^3$.

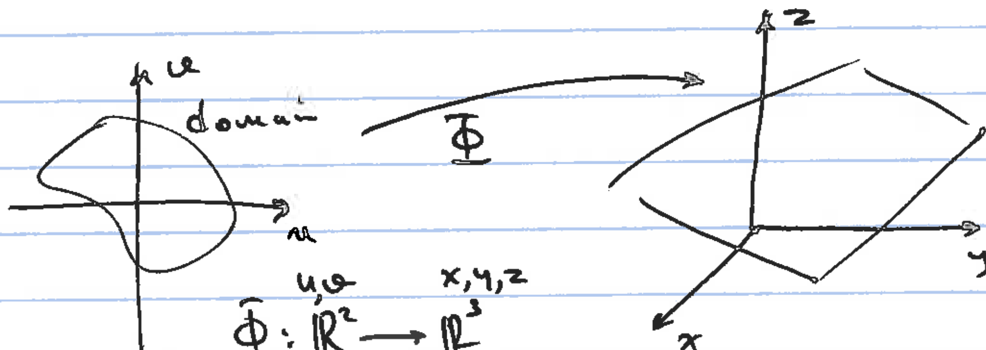
① $f(x,y)=z \; : \; \underset{x,y}{\mathbb{R}^2} \longrightarrow \underset{z}{\mathbb{R}^1}$.

$\{(x,y,z) \mid z = f(x,y)\}$

(Explicit) Graph
$\subseteq$ Domain $\times$ Range



②



$\Phi : \underset{u,v}{\mathbb{R}^2} \longrightarrow \underset{x,y,z}{\mathbb{R}^3}$

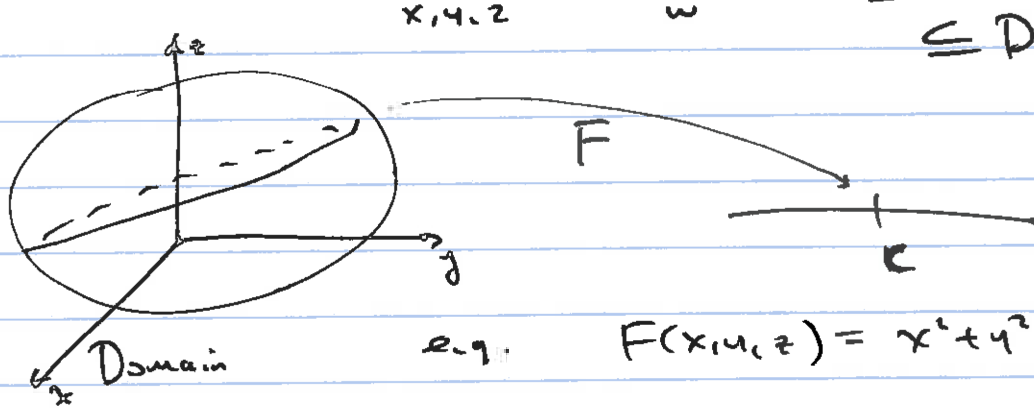
e.g. $\Phi(u,v) = (1,1,1) + u(0,1,0) + v(0,1,1)$

Parametric graph $\subseteq$ Range

③ $w = F(x,y,z) \; : \; \underset{x,y,z}{\mathbb{R}^3} \longrightarrow \underset{w}{\mathbb{R}^1}$

<u>Implicit graphs</u>
$\subseteq$ Domain.

$F^{-1}(c)$
$\parallel$
$\{(x,y,z) \mid F(x,y,z) = c\}$



e.g. $F(x,y,z) = x^2+y^2+z^2 = c$

Thm: Let $M \subseteq \mathbb{R}^3$. The following are equivalent. (Needs to be done in C^1 category)

(P) M is locally a regular parametric graph about every pt $p \in M$

(E) M is locally a graph (explicit graph) about every pt p of M

$$z = g_1(x,y)$$
or
$$x = g_2(y,z)$$
or
$$y = g_3(x,z)$$

(I) M is locally an implicit graph about every point p of M, i.e. sol'n of $F(x,y,z) = c$, where c is a regular value

see page 3

Easy ones:

(E) $\Rightarrow$ (I)

(Ex) Explicit graph of

$$z = f(x,y) = e^{xy} + \cos xy$$

is the same as

Implicit graph of $F(x,y,z) = z - (e^{xy} + \cos xy)$.

$$\{(x,y,z) \mid z - (e^{xy} + \cos xy) = 0\}$$

$$\underbrace{\{(x,y,z) \mid z = f(x,y)\}}_{\text{explicit}} \qquad \underbrace{\{(x,y,z) \mid F(x,y,z) = 0\}}_{\text{implicit}}$$

$$F(x,y,z) = z - f(x,y).$$

E $\Longrightarrow$ P

$z = f(x,y)$ explicit graph

$\Psi(u,v) = (u, v, f(u,v))$ parametrizes the explicit graph of $z = f(x,y)$.

Defn Let $F: U^{open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$.

we call $p_0 \in U$ a regular pt if $\nabla F(p_0) \neq 0$

a critical pt if $\nabla F(p_0) = 0$.

A value $c \in \mathbb{R}$ is called a regular value, if $F^{-1}(c) = \{\vec{p} \in U \mid F(p) = c\}$ consists of all regular pts.

Ex $F(x,y) = xy$

$\nabla F = (y, x)$

$\nabla F \neq (0,0)$ if $(x,y) \neq (0,0)$

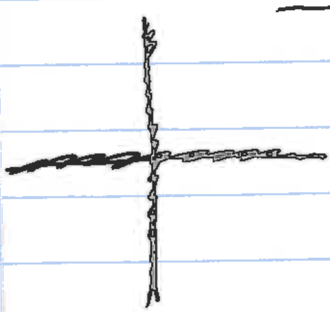
All pts of $\mathbb{R}^2 - \{(0,0)\}$ are regular

$(0,0)$ is the only critical pt. } in the Domain

points $\in$ Domain
value $\in$ Range

All $c \neq 0$, are regular values (in the Range

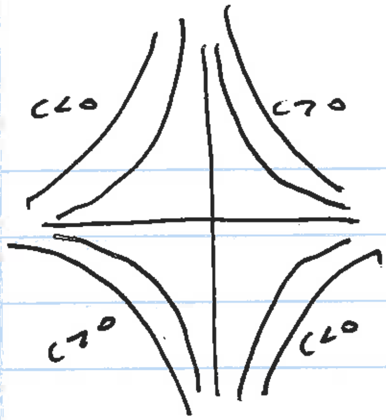
<u>$c = 0$ is not a regular value</u>



$F^{-1}(0) =$ union of x, y axes.

but only $(0,0)$ critical pt

rest of $F^{-1}(0,0)$ is all regular pts.



$F^{-1}(c) = \{(x,y) \mid xy = c\}$ $\quad c \neq 0$

I → E → P

Prop Let $F : U^{\text{open}} \subseteq \mathbb{R}^3 \xrightarrow{C^1} \mathbb{R}^1$, Let $a \in F(U)$ be a regular value, then $F^{-1}(a)$ is a regular surface.

Sketch the idea of the proof.

$$F(x,y,z) = a$$

$$DF = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}(p) \neq [0, 0\ 0]$$

where $F(p) = a$.

At least one of $\begin{cases} F_x(p) \\ F_y(p) \\ F_z(p) \end{cases}$ is not 0.

See Last page for Imp. F. Thm.

Say $F_z(p) \neq 0 \xRightarrow{\text{Imp F. Thm}} \exists\ z = g(x,y)$.

which solves

$$F(x,y,g(x,y)) = a$$

and

$$Dg = \frac{-1}{F_z}[F_x \ \ F_y].$$

So E holds. So P holds.

$\Downarrow$

$$\varphi(u,v) = (u,v,g(u,v))$$

Ex Show that

$M = \{(x,y,z) \mid x^2 - y^3 z + x^3 z^2 = 3\}$

is a regular surface.

Let $F(x,y,z) = x^2 - y^3 z + x^3 z^2$

$\nabla F = (2x + 3x^2 z^2, -3y^2 z, -y^3 + 2x^3 z)$

$\cdot \nabla F = 0$

$\cdot\ \nabla F = 0$

$M = F^{-1}(3)$

Want to show the points $\nabla F = 0$ are not on M.

$\Rightarrow$ 3 is a regular value

$\Rightarrow$ $\underset{F^{-1}(3)}{\overset{\shortparallel}{M}}$ is a 2-surface.

Suppose there is a pt on M s.t. $\nabla F = 0$ at that pt.

· We need a soln of

$$\nabla F = 0 \begin{cases} 2x + 3x^2 z = 0 & ① \\ -3y^2 z = 0 & ② \\ -y^3 + 2x^3 z = 0 & ③ \end{cases}$$

$$M \quad \} \quad x^2 - y^3 z + x^3 z^2 = 3. \quad ④$$

To show there is no common solution of ① – ④.

Case 1

② $\Rightarrow$ $y = 0$

③ $\Rightarrow$ $2x^3 z = 0$

Case 1a: $x = 0$

④ $\Rightarrow$ $3 = 0$ X

Case 1b: $z = 0$

① $\Rightarrow$ $x = 0$

④ fails.

OR

Case 2

$z = 0$

① $\Rightarrow$ $x = 0$

④ $\Rightarrow$ $0 = 3$

X

INVERSE FUNCTION THEOREM

Let $F: U_o^{open} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuously differentiable function on U_o, $\vec{a} \in U_o$, $b = F(a)$ and $F'(a)$ be an $n \times n$ invertible matrix, i.e. $\det(F'(a)) \neq 0$. Then there exists open sets U, V and a continuously differentiable function $G: V \longrightarrow U$, satisfying $a \in U \subseteq U_o$, $b \in V \subseteq \mathbb{R}^n$, $G = (F|U)^{-1}$. For all $y \in V$, $G'(y) = F'(G(y))^{-1}$.

IMPLICIT FUNCTION THEOREM

Let m equations in $n+m$ variables, $F(x_1, \ldots x_n, y_1, \ldots y_m) = \vec{0}$ be given such that

(1) $F: U_0^{open} \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ is continuously differentiable

(2) $F(a,b) = 0$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and

(3) $\det F_y(a,b) \neq 0$, where

$$F'(x,y) = \underbrace{\begin{bmatrix} F_x \end{bmatrix}}_{n \text{ columns}} \!\!\! \underbrace{\begin{bmatrix} F_y \end{bmatrix}}_{m \text{ columns}} \Big\} m \text{ rows}$$

THEN

There exists open sets U, V, and a function $G: U \longrightarrow V$ s.t.

(a) $a \in U^{open} \subseteq \mathbb{R}^n$, $b \in V^{open} \subseteq \mathbb{R}^m$,

(b) $G(a) = b$, G is continuously differentiable, and

(c) $F(x, G(x)) \equiv 0$ $\forall x \in U$, that is, G solves $y_1, \ldots y_m$ in terms of $x_1, \ldots x_n$, near $(\vec{a}, \vec{b})$.

(d)

$$G'(\vec{x}) = -F_y^{-1}(\vec{x}, G(\vec{x})) \cdot F_x(\vec{x}, G(\vec{x}))$$

(e) G is unique satisfying (c).