2.0

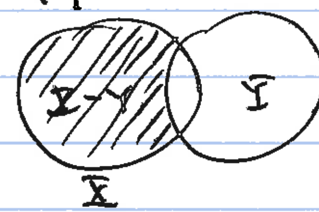
Chap II (Review of standard terminology)

Caution: center p

Defn. $B_r(p) = \{ \vec{x} \in \mathbb{R}^n \mid |\vec{x} - \vec{p}| < r \}$

open disc / open ball centered at p, of radius r.

- $X - Y = \{ \vec{x} \in X \mid \vec{x} \notin Y \}$



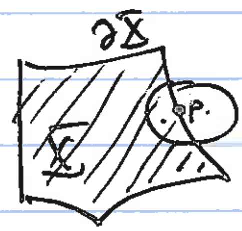
- Let $X \subseteq \mathbb{R}^n$.

  p is said to be on the boundary of X

  if $\forall r > 0$

  $B_r(p) \cap X \neq \emptyset$, and

  $B_r(p) \cap (\mathbb{R}^n - X) \neq \emptyset$.

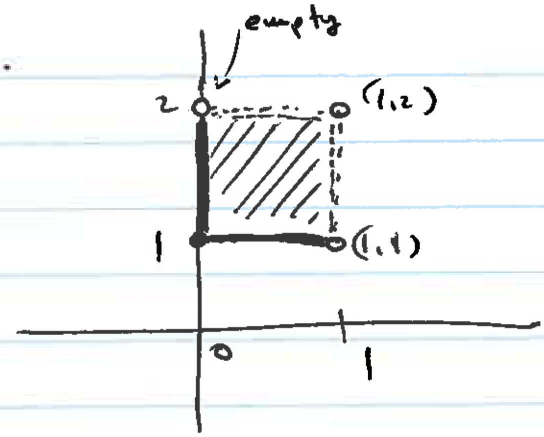


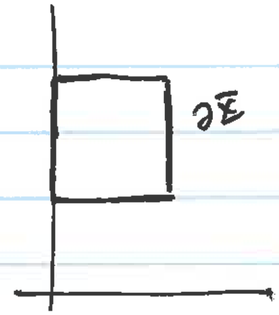
∂X denotes the set of all boundary pts of X.

- X is said to be open (in $\mathbb{R}^n$) ($X \subseteq \mathbb{R}^n$) if $\partial X \cap X = \emptyset$.

- X is said to be closed (in $\mathbb{R}^n$) if $\partial X \subseteq X$.

Ex.



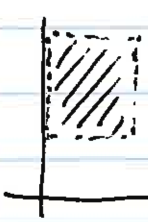


$$X = \{(x,y) \mid 0 \le x < 1,\ 1 \le y < 2\} \subseteq \mathbb{R}^2$$

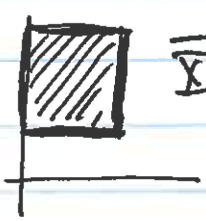
$$\partial X = \{(x,y) \mid (0 \le x \le 1 \text{ and } y = 1)$$
$$\text{or } (0 \le x \le 1 \text{ and } y = 2)$$
$$\text{or } (x = 0 \text{ and } 1 \le y \le 2)$$
$$\text{or } (x = 1 \text{ and } 1 \le y \le 2)\}$$

X is not open in $\mathbb{R}^2$
X is not closed in $\mathbb{R}^2$

Defn: $X^\circ = X - \partial X$ is called the interior of X
$\overline{X} = X \cup \partial X$ is called the closure of X



X°



$\overline{X}$

Example Continue
The above example $\operatorname{int}(X) = \{(x,y) \mid 0 < x < 1,\ 1 < y < 2\} = X^\circ$

Closure $\overline{X} = \{(x,y) \mid 0 \le x \le 1,\ 1 \le y \le 2\}$

Ex

Empty set
$\emptyset$ and $\mathbb{R}^n$ are the only sets in $\mathbb{R}^n$ which are both open and closed in $\mathbb{R}^n$.

Defn A set $\mathbb{X} \subseteq \mathbb{R}^n$ is called bounded if $\exists R$ s.t.

$$\mathbb{X} \subseteq B_R(\vec{0})$$

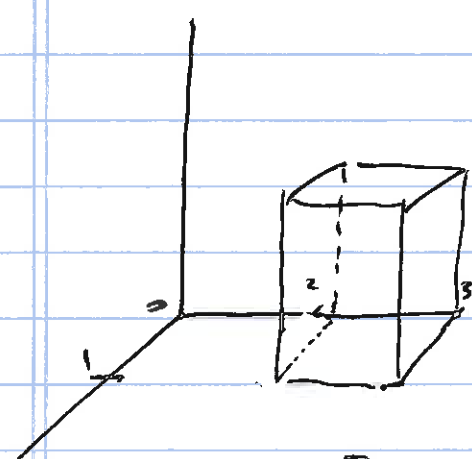
i.e $\exists R$ s.t. $\forall x \in \mathbb{X}$, $|x| < R$.

(via Heine-Borel Thm) **Defn** A set $\mathbb{X} \subseteq \mathbb{R}^n$ is called compact in $\mathbb{R}^n$ if (i) $\mathbb{X}$ is closed, and
(ii) $\mathbb{X}$ is bounded in $\mathbb{R}^n$.

Ex $[a,b]$ compact in $\mathbb{R}^1$

$\{(x,y) \mid x^2+y^2 \le 5\}$ is compact in $\mathbb{R}^2$.

$$[0,1]\times[2,3]\times[0,2] = \{(x,y,z) \mid 0 \le x \le 1 \ \& \ 2 \le y \le 3 \ \& \ 0 \le z \le 2\}$$



Compact in $\mathbb{R}^3$.

Ex (i) $\mathbb{X} = [0, \infty) \subseteq \mathbb{R}^1$ $\quad \partial\mathbb{X} = \{0\} \subseteq \mathbb{X}$.
$\mathbb{X}$ closed, unbounded, $\mathbb{X}$ not compact

(ii) $\mathbb{Y} = (0,1) \subseteq \mathbb{R}^1$ bounded,
not closed: $\partial\mathbb{Y} = \{0,1\}$, $\partial Y \nsubseteq Y$ $\Big\}$ not compact

Defn A $f : \underset{\text{open}}{X} \subseteq \mathbb{R}^n \longrightarrow \underset{\text{open}}{Y} \subseteq \mathbb{R}^n$ is

- called a homeomorphism if
  (i) f is 1-1 and onto, hence f^{-1} exists, and
  (ii) both f and f^{-1} are continuous.

- called a diffeomorphism if
  (i) f homeomorphism and
  (ii) both f and f^{-1} are C^1, and
  $\det(Df) \neq 0$, $\det(D(f^{-1})) \neq 0$.

Defn $f : X^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$f(x^1, x^2, \dots x^n) = (f^1, f^2, \dots f^m)$$

$$Df = f' = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^3} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & & \frac{\partial f^2}{\partial x^n} \\ \vdots & & & & \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & & \frac{\partial f^m}{\partial x^n} \end{bmatrix}$$

$m \times n$ matrix.

Ψ, Φ functions rather than $\vec{x}$ boldface in the book.
$\emptyset$ empty set

2.1

Def

- A function $\Phi : D^{open} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is called regular if: $(u,v) \quad (x,y,z)$

(i) $\Phi \in C^\infty$ (smooth function)

(ii) $\Phi_u \times \Phi_v = \dfrac{\partial \Phi}{\partial u} \times \dfrac{\partial \Phi}{\partial v} \neq 0$ on D.

$$\left(\Longleftrightarrow \quad D\Phi = \Phi' = \underbrace{\begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}}_{3\times 2} \quad \text{has rank 2} \right)$$

- Φ is called a patch/parametrization if (i) Φ is regular, and (ii) Φ is 1-1.

Ex $\Phi(u,v) = (u^2, v, u-v) : (0,1)\times(1,5) \to \mathbb{R}^3$

$$\begin{array}{l} \qquad\quad\;\; i \quad\; j \quad k \\ \Phi_u = (\,2u,\ 0,\ 1\,) \\ \Phi_v = (\,0,\ 1,\ -1\,) \end{array} \qquad \Phi_u \times \Phi_v = (-1, 2u, 2u) \neq \vec{0}.$$

$$D\Phi = \begin{bmatrix} 2u & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \underline{\text{rank 2}} \ \text{since} \ \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \neq 0$$

Φ 1-1 ?

$$\Phi(u_1, v_1) = \Phi(u_2, v_2) \overset{\text{WTS}}{\Longrightarrow} (u_1, v_1) = (u_2, v_2)$$

$$\Phi(u_1, v_1) = (u_1^2,\ v_1,\ u_1 - v_1)$$

$$\| \qquad\qquad \updownarrow (1) \qquad \updownarrow (2)$$

$$\Phi(u_2, v_2) = (u_2^2,\ v_2,\ u_2 - v_2)$$

(1) $\Longrightarrow v_1 = v_2$

(2) $\Longrightarrow u_1 - v_1 = u_2 - v_2$

$$v_1 = v_2$$

$$u_1 = u_2 \underbrace{- v_2 + v_1}_{0}$$

$$u_1 = u_2$$