<u>Exc</u> 1.4.7

$\beta(t) = e^t(\cos t, \sin t, 1)$ Find T, N, B, κ, τ

$C = \cos t$

$S = \sin t$

$\beta = e^t(C, S, 1)$

$\beta' = e^t(C, S, 1) + e^t(-S, C, 0)$

$= e^t(C-S, S+C, 1)$

$\beta'' = e^t(C-S, S+C, 1) + e^t(-S-C, C-S, 0)$

$= e^t(-2S, 2C, 1)$

$\beta''' = e^t(-2S, 2C, 1) + e^t(-2C, -2S, 0)$

$= e^t(-2S-2C, 2C-2S, 1)$

$v = |\beta'(t)| =$

$= e^t\sqrt{(C-S)^2 + (C+S)^2 + 1}$

$= e^t\sqrt{C^2 - \cancel{2CS} + S^2 + C^2 + \cancel{2CS} + S^2 + 1}$

$v = e^t\sqrt{3}$

$$T = \frac{\beta'}{v} = \frac{e^t(c-s,\ s+c,\ 1)}{e^t\sqrt{3}}$$

$$T = (c-s,\ s+c,\ 1)/\sqrt{3}$$

$$\kappa = \frac{|\beta' \times \beta''|}{|\beta'|^3}$$

$$\beta' \times \beta'' = \begin{vmatrix} i & j & k \\ c-s & s+c & 1 \\ -2s & 2c & 1 \end{vmatrix} e^t \cdot e^t$$

$$= e^{2t}(s+c-2c,\ -2s-c+s,$$

$$2c^2-2cs+2cs+2s$$

$$= e^{2t}(s-c,\ -s-c,\ 2)$$

$$|\beta' \times \beta''| = e^{2t}\sqrt{s^2-\cancel{2sc}+c^2+s^2+\cancel{2sc}+c^2+4}$$

$$= e^{2t}\sqrt{6}$$

$$\kappa = \frac{e^{2t}\sqrt{6}}{(e^t\sqrt{3})^3} = \frac{e^{2t}}{e^{3t}}\ \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3e^t}$$

$$\frac{\beta' \times \beta''}{|\beta' \times \beta''|} = B = (s-c,\ -s-c,\ 2) \cdot \frac{1}{\sqrt{6}}$$

$$\kappa N v = T' = \frac{d}{dt}\left((c-s,\ s+c,\ 1)\frac{1}{\sqrt{3}}\right)$$

$$= (-s-c,\ c-s,\ 0)\frac{1}{\sqrt{3}}$$

$$\kappa = \frac{\sqrt{2}}{3e^t} \qquad v = e^t\sqrt{3} \qquad \kappa v = \sqrt{\frac{2}{3}}$$

$$\sqrt{\frac{2}{3}}\,N = (-s-c,\ c-s,\ 0)\frac{1}{\sqrt{3}}$$

$$N = \frac{1}{\sqrt{2}}(-s-c,\ c-s,\ 0)$$

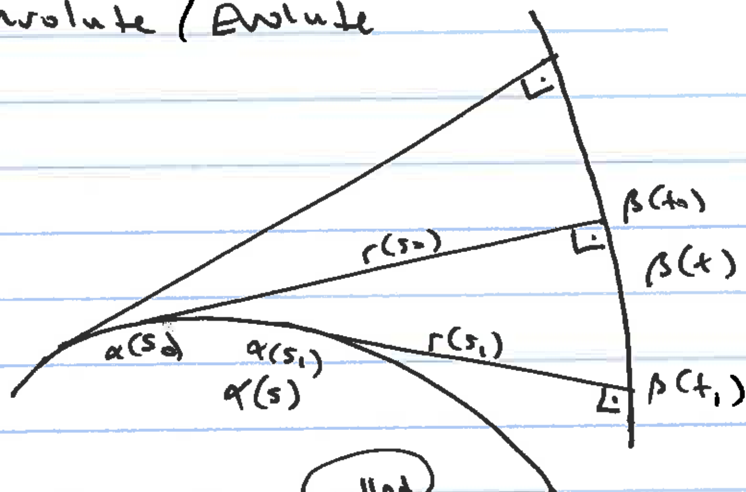
$$-\tau N v = B' = \frac{d}{dt}(s-c,\ -s-c,\ 2)\frac{1}{\sqrt{6}}$$

$$= (c+s,\ -c+s,\ 0)\frac{1}{\sqrt{6}}$$

$$-\tau\left(\frac{1}{\sqrt{2}}(-s-c,\ c-s,\ 0)\right)\cdot e^t\sqrt{3} = (c+s,-s+c,0)\frac{1}{\sqrt{6}}$$

$$\tau = \frac{1}{3e^t}$$

Classical defn.
Involute / Evolute



WANT:
$\alpha'(s_0) \perp \beta'(t_0)$
$s_0 \leftrightarrow t_0$

Defn Let $\alpha(s)$, $\beta(t)$ be regular curves & $t = t(s)$. If $\beta(t(s))$ is on the tangent line to α at $\alpha(s)$, and $\beta'(t(s)) \perp \alpha'(s)$,

then β and α are called an involute – evolute pair.

Caution, these pairings are not 1-1. (See below) ⊛ ⊛⊛

WLOG assume $|\alpha'(s)| = 1$

$$I_\alpha = \beta(s) = \alpha(s) + r(s)\,T_\alpha(s)$$

$$\beta' = T_\alpha + r' T_\alpha + r \kappa_\alpha N_\alpha$$

$$= (1 + r')\,T_\alpha + r\kappa_\alpha N_\alpha$$

$$\left.\begin{array}{l}\beta'(s) \perp \alpha'(s) \\ T_\alpha \perp N_\alpha\end{array}\right\} \Longrightarrow \quad 1 + r' = 0$$

$$r' = -1$$

$$r = -s + c_0$$

$$I_\alpha = \beta(s) = \alpha(s) - (s - c_0)\,T_\alpha(s)$$

c_0 only affects the starting pt of the involute at $\alpha(c_0)$

$\Longrightarrow \exists$ only many involutes of α.
⊛

Q: What if α is not parametrized wrt arc length?

$$0 \neq |\alpha'(t)| \not\equiv 1.$$

$$I_\alpha = \alpha(t) - (s(t) - c_0)\frac{\alpha'(t)}{|\alpha'(t)|}$$

A: Involute of a reparametrization (of α in the same direction) is a reparametr. of I_α.

Ex 1.4.11 (Caution/compare to the book)

In $\mathbb{R}^2$ $\tau = 0$, $\kappa' \neq 0$ i.e. $\kappa \nearrow$ or $\kappa \searrow$

① one defines $\mathcal{E}_\beta = \beta + \frac{1}{\kappa_\beta} N_\beta$ (if $\frac{1}{\kappa}\nearrow$)

We want to see involute of the evolute is the curve in the beginning:

$$\mathcal{E}_\beta' = \beta' + \left(\frac{1}{\kappa_\beta}\right)' N_\beta + \left(\frac{1}{\kappa_\beta}\right) N_\beta'$$

$$= T_\beta v_\beta + \left(\frac{1}{\kappa_\beta}\right)' N_\beta + \frac{1}{\kappa_\beta} v_\beta\left(-\kappa_\beta T_\beta + \cancel{\tau_\beta}^{\,0} B_\beta\right)$$

$$= \mathcal{E}_\beta' = \left(\frac{1}{\kappa_\beta}\right)' N_\beta. \text{ Hence}$$

② $v_{\mathcal{E}_\beta} = \left(\frac{1}{\kappa_\beta}\right)' \geq 0 \qquad T_{\mathcal{E}_\beta} = N_\beta$

$$S_{\varepsilon_\beta}(t) = \int_a^t \left(\frac{1}{k_\beta}\right)' dt = \frac{1}{k_\beta}(t) - \underbrace{\frac{1}{k_\beta}(a)}_{\text{choose } -c_0}$$

③ $$S_{\varepsilon_\beta}(t) = \frac{1}{k_\beta}(t) + c_0$$

$$I_{\varepsilon_\beta}(t) = \varepsilon_\beta(t) + \left(c_0 - s_{\varepsilon_\beta}(t)\right) T_{\varepsilon_\beta}(t)$$

① ③ ②

$$= \beta(t) + \frac{1}{k_\beta(t)} N_\beta(t) - \frac{1}{k_\beta(t)} N_\beta(t)$$

$$= \beta(t)$$

✲✲ In $\mathbb{R}^3$ Many curves may have the same involute.

Recall your HW problem 1.2.8
The involute of $\alpha = (R\cos wt, R\sin wt, ht)$

is

$$I_\alpha = \left(R(\cos wt + wt \sin wt), R(\sin wt - wt\cos wt), 0\right)$$

This formula shows that there is no effect of h in $I_\alpha(t)$.
different h's will give different curves α when w, R are fixed, but they all have the same involute.