1.4 Let $\alpha(t)$ be a regular curve, of class C^3.

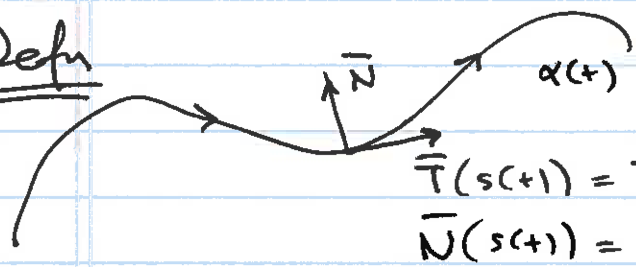
$\exists$ $\bar{\alpha}(s)$ an arclength parametrization of $\alpha(t)$

$$\bar{\alpha}(s(t)) = \alpha(t).$$

From 1.3, There exist $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}$ for $\bar{\alpha}$.

since $|\bar{\alpha}'(s)| \equiv 1$.
provided that $\bar{\kappa} > 0$.

Defn



$$\left.\begin{aligned} \bar{T}(s(t)) &= T(t) \\ \bar{N}(s(t)) &= N(t) \\ \bar{B}(s(t)) &= B(t) \\ \bar{\kappa}(s(t)) &= \kappa(t) \\ \bar{\tau}(s(t)) &= \tau(t) \end{aligned}\right\} \text{ need } \bar{\kappa} > 0$$

Notation $|\alpha'(t)| = v(t) = \text{speed} = \frac{ds}{dt}$

Obs: $T = \frac{\alpha'}{v}$ since $\frac{d}{dt}\alpha(t) = \frac{d}{dt}\bar{\alpha}(s(t)) = \frac{d\bar{\alpha}}{ds}\cdot\frac{ds}{dt} = Tv$

Thm:

$$\frac{d}{dt}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Proof (Will do only one, the rest are similar)

$$\frac{d}{dt}T(t) = \frac{d}{dt}\bar{T}(s(t)) = \frac{d\bar{T}}{ds}\cdot\frac{ds}{dt}$$

$$= \bar{\kappa}(s(t))\cdot\bar{N}(s(t))\cdot v(t) = \kappa(t)N(t)\cdot v(t)$$

Lemma: Let $\alpha : I \longrightarrow \mathbb{R}^3$, regular, C^3. Then
$\alpha = \alpha(t)$

(i) $\alpha' = vT$

(ii) $\alpha'' = v'T + \kappa v^2 N$

(iii) $\alpha''' = (v'' - \kappa^2 v^3)T + (3\kappa v v' + v^2\kappa')N + v^3\kappa\tau B$.

Proof:

(i) $\alpha' = \dfrac{d\alpha}{dt} = \dfrac{d\bar{\alpha}}{ds}\cdot\dfrac{ds}{dt} = \bar{T}(s(t))\cdot v(t) = T(t)v(t)$

$\alpha' = Tv$

(ii) $\alpha'' = T'v + Tv'$

$= (v\kappa N)v + Tv'$

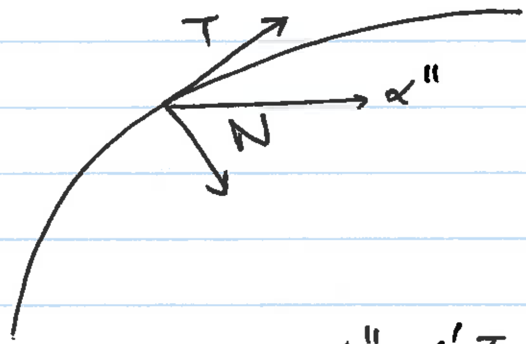
$= \kappa v^2 N + Tv'$

(iii) $\alpha''' = \kappa' v^2 N + \kappa 2vv' N + \kappa v^2 N' + T'v' + Tv''$

$= \kappa' v^2 N + 2\kappa v v' N + \kappa v^2(-\kappa T + \tau B)v + \dots$

$\dots + v'v\kappa N + Tv''$

$= T(-\kappa^2 v^3 + v'') + N(3\kappa v v' + \kappa' v^2) + B(v^3\kappa\tau)$



$\alpha' = vT$

velocity = speed · direction

$$\alpha'' = \underbrace{v'T}_{\substack{\text{Tangential}\\ \text{acceleration}}} + \underbrace{\kappa v^2 N}_{\substack{\text{normal}\\ \text{acceleration}}}$$

Corollary of Lemma: For a regular C^3 curve with $\kappa > 0$

i)
$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

ii)
$$B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$$

iii)
$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

Caution if $\alpha''(t_0) = 0$ OR $(\alpha' \times \alpha'')(t_0) = 0$, then $\kappa(t_0) = 0$ (1) still holds but (2)(3) fail.

Proof: (i) $\alpha' = vT$

$\alpha'' = v'T + \kappa v^2 N$

$$\alpha' \times \alpha'' = (vT) \times (v'T + \kappa v^2 N)$$

$$= vv'\underbrace{(T \times T)}_{0} + \kappa v^3 \underbrace{(T \times N)}_{B} = \kappa v^3 B$$

$=$

$|\alpha' \times \alpha''| = \kappa v^3$

$|\alpha'| = v$

$$\frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{\kappa v^3}{v^3} = \kappa .$$

(ii) $$\frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = \frac{\kappa v^3 B}{\kappa v^3} = B .$$

$\kappa > 0$

$v > 0$

(iii) $$\frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2} = \frac{\kappa v^3 B \cdot (* T + ** N + v^3 \kappa \tau B)}{(\kappa v^3)^2}$$

$B \cdot N = B \cdot T = 0$

$$= \frac{(\kappa v^3)(\kappa v^3)\tau \cdot 1}{(\kappa v^3)^2} = \tau .$$

Prop: If $\alpha : I \longrightarrow \mathbb{R}^n$, of class C^2 then

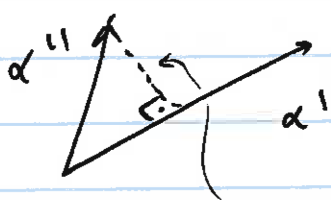
$$\kappa(t) = \frac{|\alpha'' \cdot (\alpha' \cdot \alpha') - (\alpha'' \cdot \alpha') \cdot \alpha'|}{|\alpha'|^4}$$

Proof: $\alpha' = Tv$ $\qquad \Big(\overset{\text{Def}^n \text{ in } \mathbb{R}^n}{T = \frac{\alpha'}{v}}$ and $v = |\alpha'| \Big)$

$$\alpha'' = T'v + Tv' = \kappa N v^2 + Tv'$$

$$N = \frac{T'}{|T'|} \leftarrow \text{Assume not } 0$$

Def^n in $\mathbb{R}^n$



Orthogonally Project α'' along α'

$$\alpha'' - \frac{\alpha'' \cdot \alpha'}{\alpha' \cdot \alpha'} \alpha'$$

$$= (\kappa N v^2 + Tv') - \frac{(\kappa N v + Tv') \cdot Tv}{Tv \cdot Tv} Tv$$

$T \cdot T = 1$
$T \cdot N = 0$

$$= (\kappa N v^2 + Tv') - \frac{vv'}{v^2} Tv$$

$$= \kappa N v^2 + Tv' - Tv'$$

$$= \kappa N v^2$$

(PTO)

$$\alpha'' - \frac{\alpha''\cdot\alpha'}{\alpha'\cdot\alpha'}\,\alpha' = \kappa N v^2$$

$$(\alpha'\cdot\alpha')\cdot\alpha'' - (\alpha''\cdot\alpha')\,\alpha' = \underbrace{(\alpha'\cdot\alpha')}_{v^2}\cdot\kappa N v^2$$

$$\left|(\alpha'\cdot\alpha')\,\alpha'' - (\alpha''\cdot\alpha')\,\alpha'\right| = v^4\kappa$$ since $|N| = 1$.

$$\kappa = \frac{\left|(\alpha'\cdot\alpha')\,\alpha'' - (\alpha''\cdot\alpha')\,\alpha'\right|}{|\alpha'|^4}$$ since $v = |\alpha'|$