Sept 11, 2017

①

Not in our textbook

§ Isometries of $\mathbb{R}^n$ and Fund. Thm of Curves

Thm: Let $f: \mathbb{R}^n \to \mathbb{R}^n$

$\forall x,y \in \mathbb{R}^n \quad |f(x) - f(y)| = |x-y|$ (Isometry)

$\iff f(x) = A \cdot x + b$ for some $\begin{cases} A \in O(n) \\ b \in \mathbb{R}^n. \end{cases}$

Defn $O(n) = \{ A \text{ } n\times n \text{ matrix} \mid AA^T = Id_n \}$

$A \in O(n) \Rightarrow \det A = \pm 1$

Rigid motions exclude reflections

$SO(n) = \left\{ A \text{ } n\times n \text{ matrix} \;\middle|\; \begin{array}{l} AA^T = Id_n \\ \det A = +1 \end{array} \right\}$

$\left. \begin{array}{l} f \text{ is a rigid motion} \\ \text{of } \mathbb{R}^n \end{array} \right\} \iff \begin{cases} f = A\cdot x + b \\ A \in SO(n) \\ b \in \mathbb{R}^n. \end{cases}$

So: ① Rigid motions of $\mathbb{R}^n$ are the compositions of all translations and rotations.

② Isometries of $\mathbb{R}^n$ are the compositions of all translations, rotations and reflections.

FUNDAMENTAL THM of CURVES

Let $I = [a,b]$, $\kappa, \tau : [a,b] \longrightarrow \mathbb{R}$, be C^1,
$\kappa(s) > 0 \quad \forall s \in I$
$\tau(s) \in \mathbb{R}$

Then $\exists$ a regular parametrized curve $\beta(s) : I \longrightarrow \mathbb{R}^3$ s.t.

1) s is the arclength of β
2) $\kappa(s)$ is the curvature of β
3) $\tau(s)$ is the torsion of β

β is unique up to a rigid motion of $\mathbb{R}^3$.
If $\tilde{\beta}$ is another curve satisfying 1–3 above then

$$\tilde{\beta}(s) = A \cdot \beta(s) + b \quad \text{for some}$$
$$A \in SO(n), \quad b \in \mathbb{R}^n.$$

Remark: Under reflections, κ stays same but τ changes sign.

Recall $\kappa \equiv 0 \iff \beta$ is a line

$\left.\begin{array}{l}\tau = 0 \\ \kappa = \kappa_0\end{array}\right\} \iff \beta$ is a part of a circle

$\begin{array}{l}\tau = \tau_0 \neq 0 \\ \kappa = \kappa_0 > 0\end{array} \quad \Longleftarrow \quad$ Helix

$(R\cos t, R\sin t, ht)$

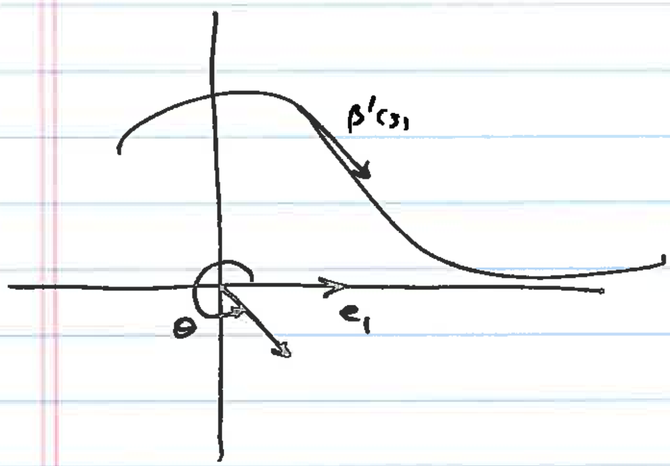
$\kappa = \dfrac{R}{R^2+h^2}$

$\tau = \dfrac{h}{R^2+h^2}$

FTCurves $\Longrightarrow \left\{ \begin{array}{l}\text{Helices are unique} \\ \text{upto rigid motions}\end{array}\right.$

Signed curvature in $\mathbb{R}^2$

Let $\beta : I \longrightarrow \mathbb{R}^2$ be a C^2 curve, $|\beta'(s)| \equiv 1$

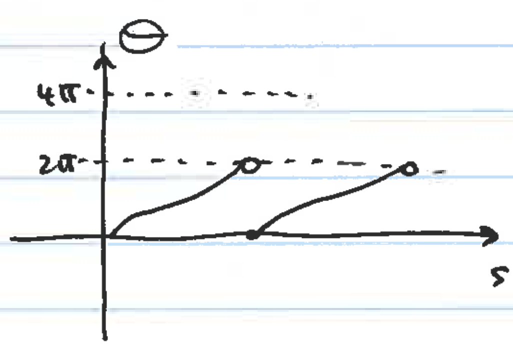


$T = \beta'(s)$ unit vector

$T = (\cos \theta(s), \sin \theta(s))$

$\theta = \measuredangle_o (e_1, T)$

oriented angle measured counter clockwise from $\vec{i} = e_1 = (1,0)$



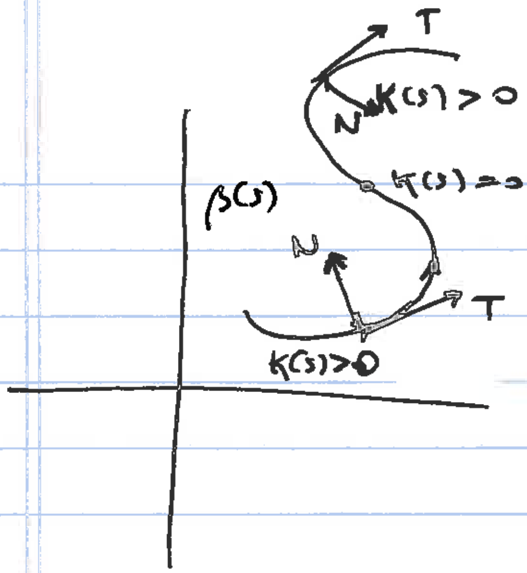
$\theta(s)$ may have jump discontinuities at $0, 2\pi$

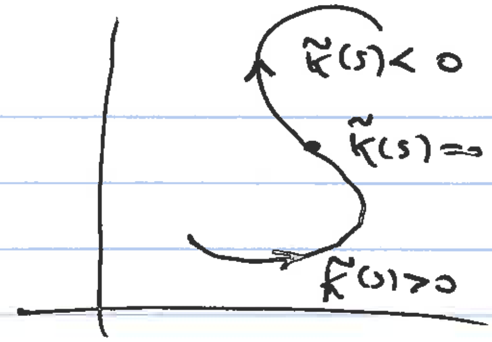
When $\theta(s) \in (0, 2\pi)$, $\theta \in C^1$.

$$\frac{dT}{ds} = \left(-\sin \theta(s) \cdot \frac{d\theta}{ds}, \ \cos \theta(s) \cdot \frac{d\theta}{ds} \right)$$

$$\kappa(s) = |\beta''(s)| = |T'(s)| = \left| \frac{d\theta}{ds} \right|$$

standard curvature.



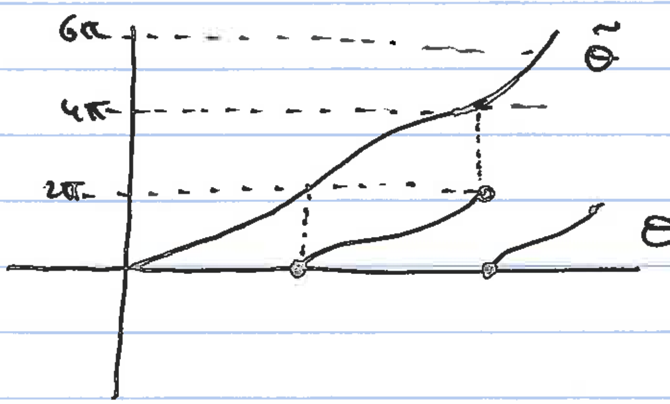


We define the signed curvature
$$\tilde{\kappa}(s) = \begin{cases} 0 & \text{if } \kappa(s)=0 \\ \kappa(s) & \text{if } \measuredangle_0(T,N)=\frac{\pi}{2} \\ -\kappa(s) & \text{if } \measuredangle_0(T,N)=\frac{3\pi}{2} \end{cases}$$

$$\tilde{\kappa}(s) = \frac{d\theta}{ds} \quad \text{when} \quad \Theta(s) \in (0, 2\pi)$$

$\measuredangle_0 (e_1, \beta'(a))$

We define
$$\tilde{\Theta}(s) = \Theta(a) + \int_a^s \tilde{\kappa}(u)\,du$$

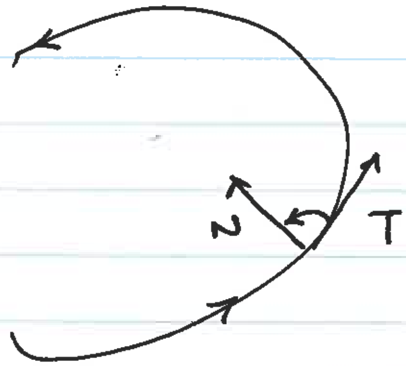


$\tilde{\Theta} \in C^1$ on all of $\underline{\underline{[a,b]}}$

$$\beta'(s) = T(s) = (\cos\tilde{\Theta}(s), \sin\tilde{\Theta}(s))$$

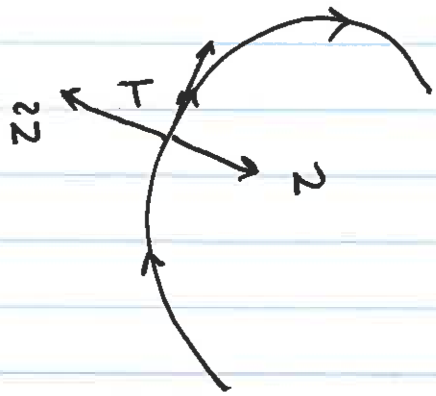
$$\tilde{\kappa}(s) = \tilde{\Theta}'(s)$$
$$\kappa(s) = |\tilde{\Theta}'(s)|$$

Case I



$\tilde{\kappa}(s) = \kappa(s) > 0$

Case 2



$0 > \tilde{\kappa}(s) = -\kappa(s)$

$$\left.\begin{aligned} T' &= \kappa N \\ T' &= \tilde{\kappa} \tilde{N} \end{aligned}\right\} \text{Both Cases}$$

$$\begin{aligned} T &= (\cos\tilde{\theta}(s), +\sin\tilde{\theta}(s)) \\ \tilde{N} &= (-\sin\tilde{\theta}(s), \cos\tilde{\theta}(s)) \\ N &= (\operatorname{sign}\tilde{\kappa}) \cdot \tilde{N} \end{aligned}$$

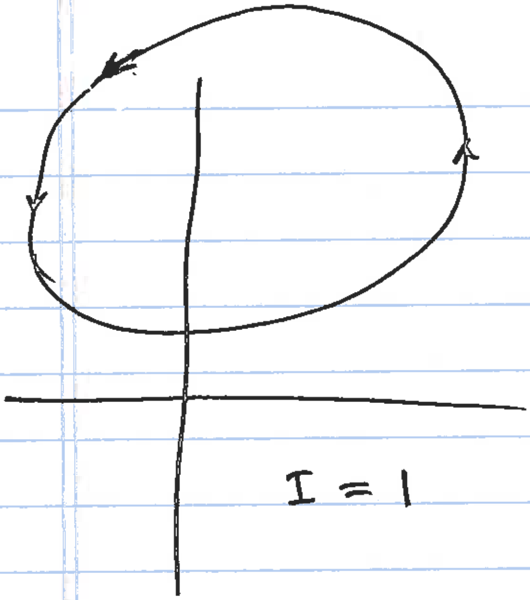
Defn Let $\beta(s)$ be a C^1 closed, C^2 curve.

$\beta : [a,b] \to \mathbb{R}^2$

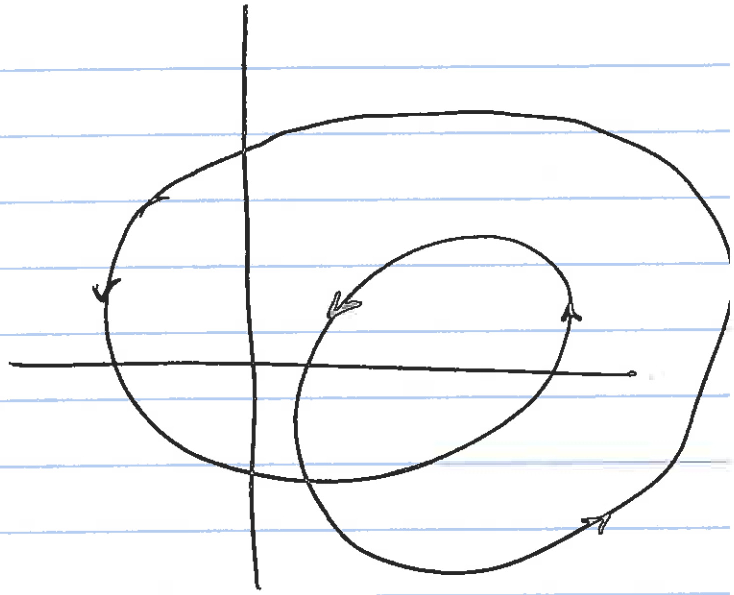
The rotation index of β is defined by

$$\frac{1}{2\pi}\int_a^b \tilde{\kappa}(s)\, ds = I(\beta).$$

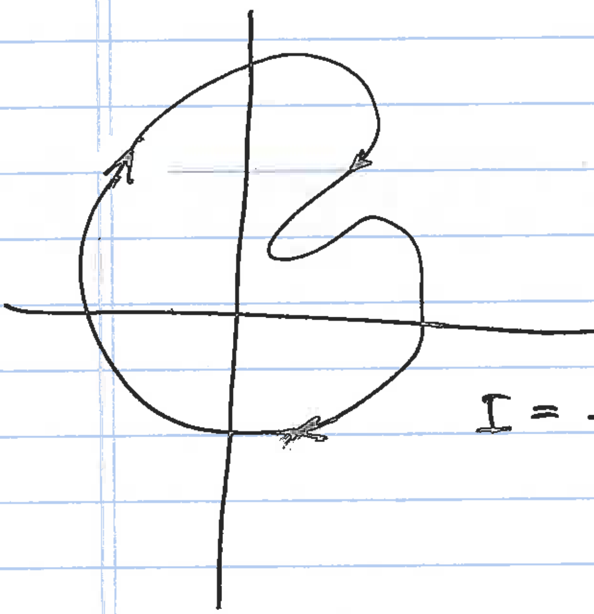
Prop $I(\beta) \in \mathbb{Z}$ for all C^1 closed, C^2 curves



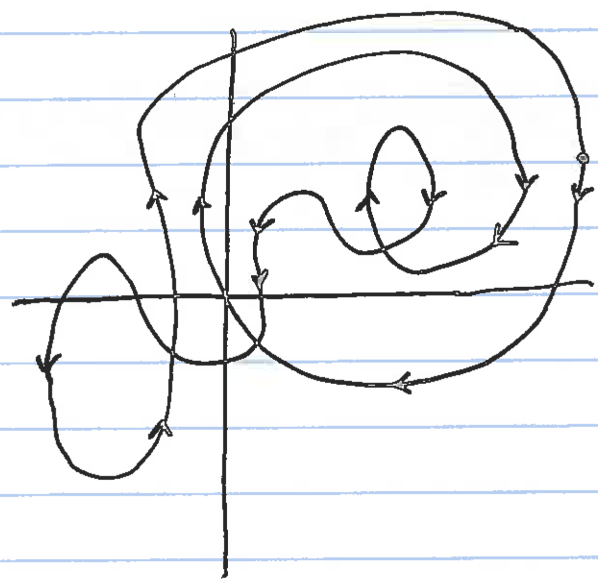
I = 1



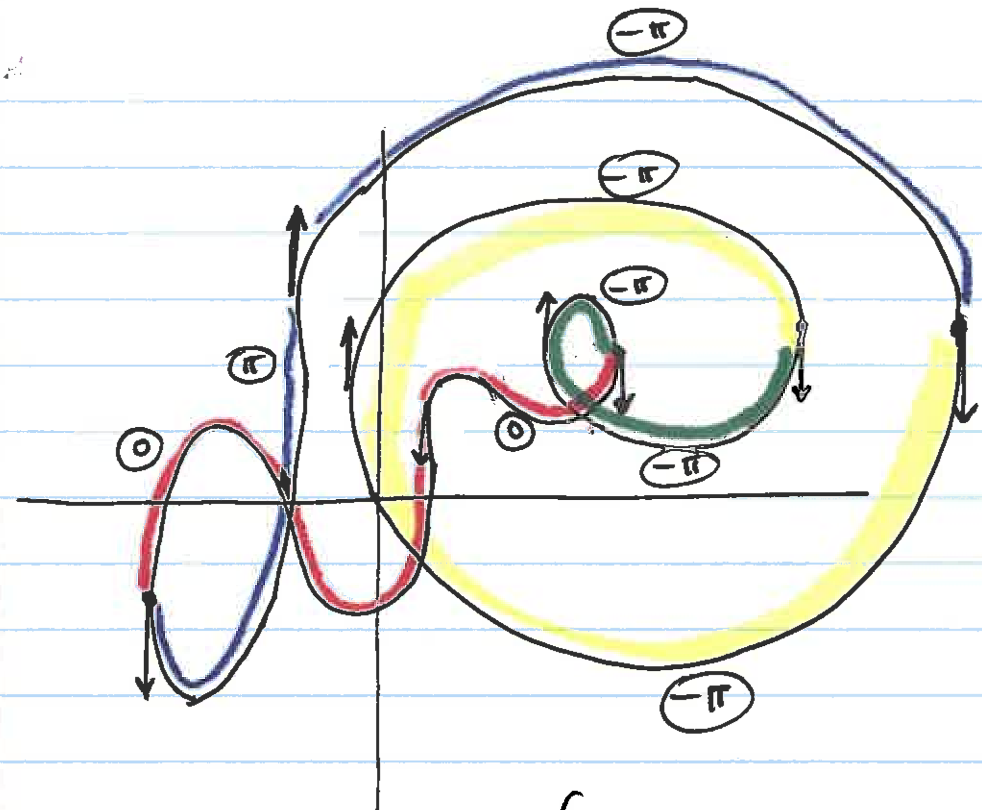
I = 2



I = −1



I = ?



$$\int \tilde{k}\, ds = -4\pi$$

$$I = -2$$