9/1/17

1.3 Continue:

β regular, param w.r.t arclength, $\beta \in C^3$

$\beta' = T$.

T', $|T'| = \kappa$, $N = \frac{T'}{\kappa}$ when $\kappa > 0$

$B = T \times N$

B' $\quad \tau = -N \cdot B'$ or $B' = -\tau N$.

Theorem: (Frenet)

Let β be a C^3 curve s.t. $|\beta'(s)| \equiv 1$
Then whenever $\kappa(s) > 0$ ($\kappa(s) = |\beta''(s)|$) one has

$$(\beta' = T)$$

$$\begin{aligned} T' &= \qquad \kappa N \\ N' &= -\kappa T \qquad\qquad + \tau B \\ B' &= \qquad -\tau N \end{aligned}$$

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof 1) $T' = \kappa N$ (Defn).

2) $B' = -\tau N$ $\Longleftarrow \begin{cases} B' /\!/ N & \text{Lemma} \\ \tau = -B'\cdot N \end{cases}$

Want:

3) $N' = -\kappa T + \tau B$.

Proof: $B = T \times N$

$N = B \times T$.

$N' = B' \times T + B \times T'$

$N' = (-\tau \underbrace{N \times T}_{-B}) + B \times (\kappa N)$

$=$

$N' = \tau B + \kappa(-T)$

Helix

Example: $\alpha(t) = (R\cos wt, R\sin wt, ht)$

Constant $R > 0$, $w \neq 0$, $h \in \mathbb{R}$

$$\alpha'(t) = (-wR\sin wt, wR\cos wt, h)$$

$$|\alpha'(t)| = \sqrt{w^2R^2(\sin^2 wt + \cos^2 wt) + h^2}$$

$$= \sqrt{w^2R^2 + h^2} > 0$$

Let $c = (w^2R^2 + h^2)^{-\frac{1}{2}}$

$$s = \int_0^t |\alpha'(u)|\,du = \int_0^t \frac{1}{c}\,du = \frac{t}{c}$$

$cs = t$

$$\beta(s) = (R\cos cws,\ R\sin cws,\ hcs)$$

$$T = \beta'(s) = (-cwR\sin cws,\ cwR\cos cws,\ hc)$$

$$|\beta'(s)| = \sqrt{c^2w^2R^2 + h^2c^2} = c\sqrt{w^2R^2+h^2} = 1.$$

$$T' = \beta''(s) = (-c^2w^2R\cos cws,\ -c^2w^2R\sin cws,\ 0)$$

$$\kappa = |T'(s)| = \sqrt{c^4w^4R^2} = c^2w^2R = \frac{w^2R}{w^2R^2+h^2}.$$

$$\left(\text{if } h=0,\ \text{i.e. circle},\ \kappa = \frac{1}{R}\right)$$

$$N = \frac{T'}{\kappa} = (-\cos cws,\ -\sin cws,\ 0)$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ -cwR\sin cws & cwR\cos cws & hc \\ -\cos cws & -\sin cws & 0 \end{vmatrix}$$

$$= (hc\sin cws,\ -hc\cos cws,\ cwR)$$

$$B' = (hc^2w\cos cws,\ hc^2w\sin cws,\ 0) = -\tau N$$

$$\left.\begin{aligned}\tau &= c^2hw = \frac{hw}{R^2w^2+h^2}.\\ \kappa &= c^2w^2R = \frac{w^2R}{w^2R^2+h^2}\end{aligned}\right\}\ \underline{\underline{\text{constant.}}}$$

* Standard Helices have constant curvature and torsion

Prop $\forall$ given $\kappa_0 > 0$, $\forall \tau_0 \in \mathbb{R}$, fixed real #'s.

$\exists$ a helix (when $\tau_0 \neq 0$)
or a circle (when $\tau_0 = 0$)

s.t. this curve has constant curvature κ_0 & constant torsion τ_0.

Proof

want
$$\begin{cases} \tau = \dfrac{h\omega}{R^2\omega^2 + h^2} \\[2ex] \kappa = \dfrac{\omega^2 R}{\omega^2 R^2 + h^2} \end{cases}$$
take $\omega = 1$

τ, κ given; Want $h, R, (\omega = 1)$

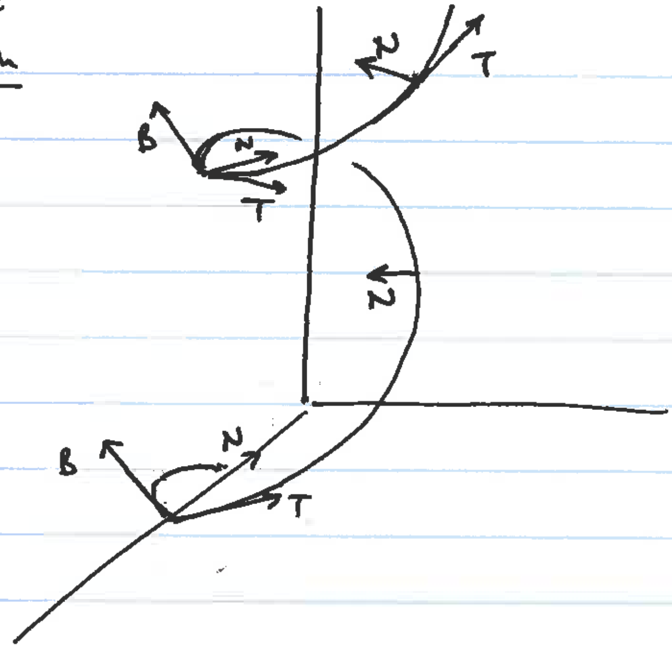
$$\left.\begin{aligned} \tau &= \frac{h}{R^2 + h^2} \\ \kappa &= \frac{R}{R^2 + h^2} \end{aligned}\right\}$$
Yes we can solve R, h in terms of κ & τ.

$$\tau^2 + \kappa^2 = \frac{h^2}{(R^2+h^2)^2} + \frac{R^2}{(R^2+h^2)^2} = \frac{1}{(R^2+h^2)}$$

$$\tau = \frac{h}{R^2+h^2} = h(\tau^2+\kappa^2) \Rightarrow h = \frac{\tau}{\tau^2+\kappa^2}$$

$$\kappa = \frac{R}{R^2+h^2} = R(\tau^2+\kappa^2) \Rightarrow R = \frac{\kappa}{\tau^2+\kappa^2}$$

Helix
Graph



N is horizontal
$\parallel$
$(*, *, 0)$
points towards
the z-axis

T slants up
B slants up but
backwards

$h > 0, \ w = 1, \ R > 0 \qquad \kappa = \frac{R}{R^2 + h^2} > 0$

Right handed helix, $\tau > 0$