Shape Operator

<u>Defn</u> Let U be unit normal to a patch of a regular surface in a neighborhood of p, on M, $v \in T_pM$. We define

$$S_p(v) = -\nabla_v U, \quad \text{shape operator}$$

Weingarten map.

<u>Obs.</u> $-\nabla_v U \in T_p\mathbb{R}^3$ a priori.

<u>Lemma: I</u> $S_p : T_pM \longrightarrow T_pM$

<u>IMPORTANT</u> <u>Obs:</u>

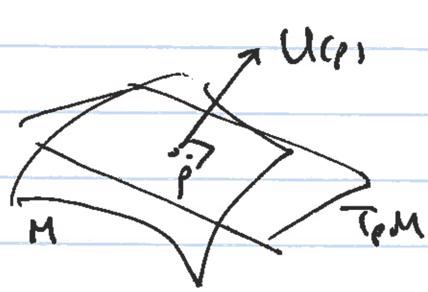
1 $$\nabla_v U(p) = \frac{d}{dt} U(\alpha(t))\Big|_{t=0}$$

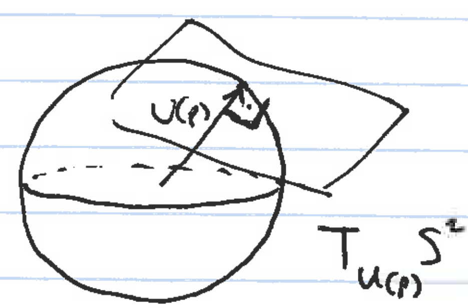
Take $\alpha(t)$ on M

$\alpha(0) = p$

$\alpha'(0) = v$

2





$U(p) \perp T_pM$

$U(p) \perp T_{U(p)}S^2$

$$\Longrightarrow T_pM = T_{U(p)}S^2$$

Proof of Lemma: I

$$U \cdot U \equiv 1.$$

$$U(\alpha(t)) \cdot U(\alpha(t)) \equiv 1$$

α is a curve $\quad \alpha(0) = p$

$\alpha'(0) = v.$

$$\frac{d}{dt}\Big(U(\alpha(t)) \cdot U(\alpha(t)) \Big) \equiv 0$$

$$\Big(\underbrace{\frac{d}{dt} U(\alpha(t))}_{\text{at } t=0} \cdot \Big) U(\alpha(t)) = 0.$$

$$(\nabla_v U)(p) \cdot \underbrace{U(p)}_{\text{normal of } T_pM.} = 0.$$

$$\Rightarrow (\nabla_v U)(p) \in T_pM.$$

Hence

$$S_p : T_pM \longrightarrow T_pM = T_{U(p)}S^2$$

Lemma II $S_p : T_pM \longrightarrow T_pM$ is a linear map.

Proof:

$$v[f] = c_1 \frac{\partial}{\partial u}(f \circ \Psi) + c_2 \frac{\partial}{\partial v}(f \circ \Psi)$$

for a parametrization Ψ of M about p.

where $v = (c_1 \Psi_u + c_2 \Psi_v)(u_0, v_0)$

$\Psi(u_0, v_0) = p$.

$$(v+w)[f] = (c_1 + d_1)\frac{\partial}{\partial u}(f \circ \Psi) + (c_2 + d_2)\frac{\partial}{\partial v}(f \circ \Psi)$$
$$= v[f] + w[f],$$

where $v = (c_1 \Psi_u + c_2 \Psi_v)(u_0, v_0)$

$w = (d_1 \Psi_u + d_2 \Psi_v)(u_0, v_0)$

$$\nabla_{v+w} U = ((v+w)[U_1], (v+w)[U_2], (v+w)[U_3])$$

where $U = (U_1, U_2, U_3)$

$$= (v[U_1], v[U_2], v[U_3]) + (w[U_1], w[U_2], w[U_3])$$

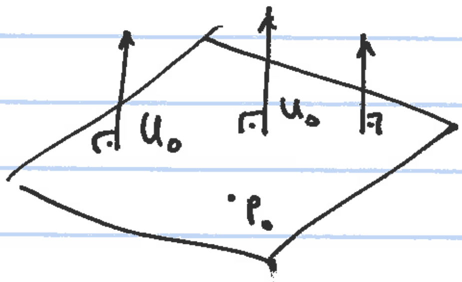
$$= \nabla_v U + \nabla_w U.$$

Similarly for $\nabla_{cv} U = c \nabla_v U \quad \forall c \in \mathbb{R}$.

Example Planes

$\forall$ plane $\exists\, U_0 \in \mathbb{R}^3$ fixed, normal vector of the given plane

$$U_0 \cdot (\vec{x} - p_0) = 0$$



$$\nabla_v U_0 \equiv 0. \quad , \quad S_p \equiv 0 \text{ for planes.}$$

IMPORTANT

Given φ a parametrization of M about $p = \varphi(u_0, v_0)$, $\{\varphi_u(u_0,v_0), \varphi_v(u_0,v_0)\}$ forms a basis of T_pM.

$$S_p : T_pM \to T_pM \quad \text{linear}$$

We want $S_p(\varphi_u)$, $S_p(\varphi_v)$

$$S_p(\varphi_u) = -\nabla_{\varphi_u} U. = -\frac{\partial}{\partial u} U \circ \varphi$$

$\varphi(u, v_0)$ coordinate curves are in the direction of φ_u.

$$S_p(\varphi_v) = -\frac{\partial}{\partial v}(U \circ \varphi)$$

Ex Sphere of Radius R.

Geographical $\psi(u,v) = (R\cos u \cos v,\ R\sin u \cos v,\ R\sin v)$

$$\psi_u = (-R\sin u \cos v,\ R\cos u\cos v,\ 0)$$
$$\psi_v = (-R\cos u \sin v,\ -R\sin u \sin v,\ R\cos v)$$

$$\psi_u \times \psi_v = N = (R^2\cos u\cos^2 v,\ R^2 \sin u \cos^2 v,\ R^2 \sin v \cos v)$$

$$R^2 \sin^2 u \sin v \cos v + R^2 \cos^2 u \cos v \sin v$$

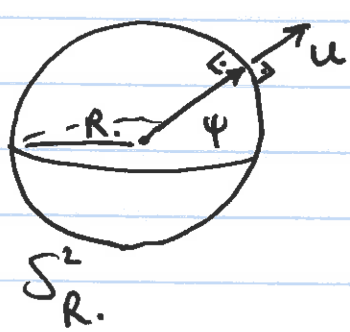
$$U = \frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|}$$

$$\left(R^4\left(\cos^2 u \cos^4 v + \sin^2 u \cos^4 v + \sin^2 v \cos^2 v\right)\right)^{1/2}$$

$$= \left(R^4\left(\underbrace{\cos^4 v}_{\cos^2 v \cdot \cos^2 v} + \sin^2 v \cos^2 v\right)\right)^{1/2}$$

$$= (R^4 \cos^2 v)^{1/2} = R^2 \cos v \geq 0$$
$$-\tfrac{\pi}{2} \leq v \leq \tfrac{\pi}{2}$$

$$U = (\cos u \cos v,\ \sin u \cos v,\ \sin v) = \frac{\psi}{R}.$$



$|\psi| = R$

$|U| = 1$

$$U = \frac{\psi}{R}.$$

$$S_p(\psi_u) = -\frac{\partial}{\partial u}(U \circ \psi) = -\frac{\partial}{\partial u}\frac{\psi}{R} = -\frac{1}{R}\psi_u.$$

$$S_p(\psi_v) = -\frac{\partial}{\partial v}(U \circ \psi) = -\frac{\partial}{\partial v}\frac{\psi}{R} = -\frac{1}{R}\psi_v$$

$$S_p : T_pM \longrightarrow T_pM$$

$$v \longmapsto -\frac{v}{R}.$$

$$[S_p]_{\{\psi_u, \psi_v\}} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}.$$

<u>Example</u> $\psi(u,v) = (u, v, u^2+v^2)$

Find the shape operator.

$$\psi_u = (1, 0, 2u)$$
$$\psi_v = (0, 1, 2v)$$

$$N = \psi_u \times \psi_v = (-2u, -2v, 1)$$

$$|\psi_u \times \psi_v| = \sqrt{4u^2+4v^2+1}$$

$$U = \frac{(-2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}} = (4u^2+4v^2+1)^{-\frac{1}{2}} \cdot (-2u, -2v, 1)$$

$$S_p(\psi_u) = -\frac{\partial}{\partial u} U = -\nabla_{\psi_u} U =$$

$$= +\frac{1}{2}(4u^2+4v^2+1)^{-\frac{3}{2}} \cdot 8u \cdot (-2u, -2v, 1) +$$
$$+ (4u^2+4v^2+1)^{-\frac{1}{2}} \cdot (2, 0, 0)$$

$$= (4u^2+4v^2+1)^{-\frac{3}{2}} \left[(-8u^2, -8uv, 4u) + (8u^2+8v^2+2, 0, 0) \right]$$

$$\underline{S_p(\psi_u)} = (4u^2+4v^2+1)^{-\frac{3}{2}} (8v^2+2, -8uv, 4u).$$

$$S_p(\psi_v) = -\frac{\partial}{\partial v} U = -\nabla_{\psi_v} U$$

$$= +\frac{1}{2}(4u^2+4v^2+1)^{-\frac{3}{2}} \cdot 8v \cdot (-2u, -2v, 1) +$$
$$+ (4u^2+4v^2+1)^{-\frac{1}{2}} \cdot (0, 2, 0)$$

$$= (4u^2+4v^2+1)^{-3/2} \Big[\underbrace{(-8uv, -8v^2, 4v) + (0, 8u^2+8v^2+2, 0)}_{(-8uv,\ 8u^2+2,\ 4v)} \Big]$$

$$S_p(\Psi_u) = (4u^2+4v^2+1)^{-\frac{3}{2}}(8v^2+2,\ -8uv,\ 4u)$$

$$S_p(\Psi_v) = (4u^2+4v^2+1)^{-\frac{3}{2}}(-8uv,\ 8u^2+2,\ 4v)$$

Now we want to write

$$S_p(\Psi_u) = a\Psi_u + b\Psi_v$$
$$S_p(\Psi_v) = c\Psi_u + d\Psi_v$$

So that $[S_p]_{\{\Psi_u,\Psi_v\}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

The matrix of S_p wrt basis $\{\Psi_u,\Psi_v\}$.

$$S_p(\Psi_u) = (4u^2+4v^2+1)^{-\frac{3}{2}}\left[\underbrace{(1,0,2u)}_{\Psi_u}\cdot(2+8v^2) + \underbrace{(0,1,2v)}_{\Psi_v}(-8uv)\right]$$

$$S_p(\Psi_v) = (4u^2+4v^2+1)^{-\frac{3}{2}}\left[(1,0,2u)\cdot(-8uv) + (0,1,2v)(2+8u^2)\right]$$

$$[S_p]_{\{\Psi_u,\Psi_v\}} = (1+4u^2+4v^2)^{-\frac{3}{2}}\begin{bmatrix} 2+8v^2 & -8uv \\ -8uv & 2+8u^2 \end{bmatrix}$$

$$\mathrm{Det}\,[S_p] = (1+4u^2+4v^2)^{-3}\cdot\left(\underbrace{\underbrace{(2+8v^2)(2+8u^2)-64u^2v^2}_{4+16u^2+16v^2+64u^2v^2-64u^2v^2}}_{4+16u^2+16v^2}\right)$$

$$\mathrm{Det}\,[S_p] = \frac{4}{(1+4u^2+4v^2)^2}$$ (Called Gaussian Curvature.)