(2.2) Example Continuing

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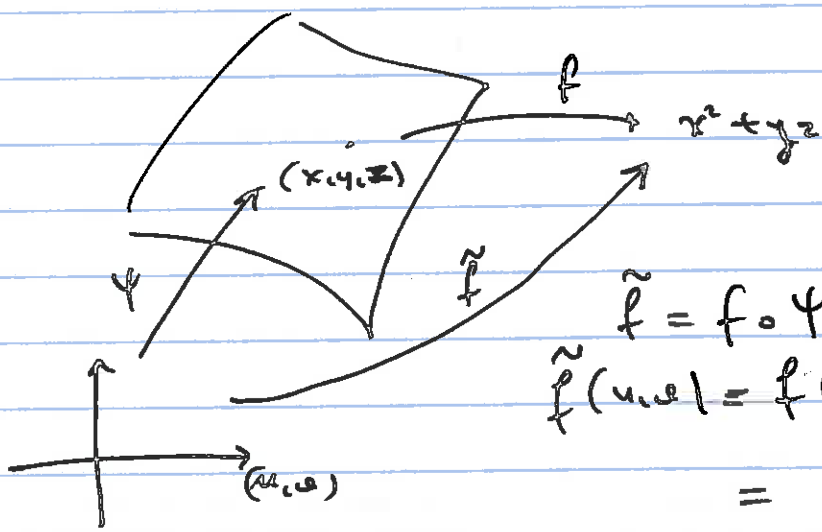
Method II

given:
$$\begin{cases} \psi(u,v) = (u,v,uv) \\ f = x^2+yz \\ p = \psi(2,3) = (2,3,6) \\ V = (2,4,14) \end{cases}$$

Want $\quad v[f](p) =$



$$\tilde{f} = f \circ \psi(u,v)$$

$$\tilde{f}(u,v) = f(u,v,uv)$$

$$= u^2 + v \cdot uv$$

$$= u^2 + uv^2$$

$$\frac{\partial \tilde{f}}{\partial u} = 2u + v^2 \qquad \frac{\partial \tilde{f}}{\partial u}(2,3) = 4+9 = 13$$

$$\frac{\partial \tilde{f}}{\partial v} = 2uv \qquad \frac{\partial \tilde{f}}{\partial v}(2,3) = 12.$$

Why? Look at pages ②, ④

$$(c_1 \psi_u + c_2 \psi_v) \cdot f = c_1 \frac{\partial \tilde{f}}{\partial u} + c_2 \frac{\partial \tilde{f}}{\partial v}$$

$$V = (2,4,14) = 2\psi_u(2,3) + 4\psi_v(2,3)$$

$$v[f](p) = 2 \cdot 13 + 4 \cdot 12 = 26 + 48 = 74.$$

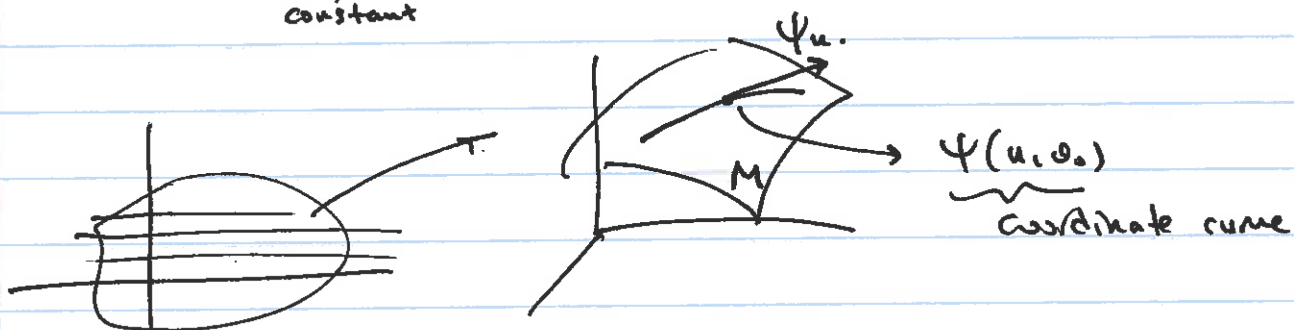
Fact $\vec{\Psi}_u \cdot f = \frac{\partial}{\partial u}(f \circ \Psi)$

Recall Defn $\frac{d}{dt} f(\alpha(t))\Big|_{t=0} = \alpha'(0) \cdot f$

$\Psi(u, v_0)$ is a curve on M

varies, constant

whose direction is $\Psi_u(u_0, v_0)$ at $\Psi(u_0, v_0)$



$$\vec{\Psi}_u[f](\Psi(u_0,v_0)) = \frac{d}{du}\left(f \circ \Psi(u, v_0)\right)\Big|_{u=u_0} = \left(\frac{\partial}{\partial u} f \circ \Psi\right)(u_0, v_0)$$

Similarly $\vec{\Psi}_v \cdot f = \frac{\partial}{\partial v}(\underbrace{f \circ \Psi}_{\tilde{f}})$

$$(c_1 \vec{\Psi}_u + c_2 \vec{\Psi}_v)(f) = \frac{\partial \tilde{f}}{\partial u} \cdot c_1 + \frac{\partial \tilde{f}}{\partial v} \cdot c_2$$

By linearity

Same Example

Method III

$$\varphi(u,v) = (u, v, uv)$$
$$f = x^2 + yz$$
$$p = \varphi(2,3) = (2,3,6)$$
$$v = (2,4,14)$$

$$v[f](p) = (\text{grad } f)(2,3,6) \cdot (2,4,14)$$
$$= (2x, z, y)\Big|_{(2,3,6)} \cdot (2,4,14)$$
$$= (4, 6, 3) \cdot (2, 4, 14)$$
$$= 8 + 24 + 42 = 74.$$

Method III will work if we know f in an open set about p. If we know f only along the surface, Method III will not work, unless we find an appropriate extention. However, most of the time, finding extension is more work.

In many cases, f will be defined along the surface only.

To Summarize

Recall <u>Def</u>

$$f : M \to \mathbb{R}$$

$p \in M$, $v \in T_pM$.

method I ⟵ for a curve $\alpha \subseteq M$, $\alpha(0) = p$, $\alpha'(0) = v$

$$\frac{d}{dt} f(\alpha(t))\Big|_{t=0} = v[f](p).$$

<u>Prop</u> (i) The definition above is independent of choice of α.

(ii) If $v = c_1 \Psi_u(u_0, v_0) + c_2 \Psi_v(u_0, v_0)$

then

method II ⟵ $$v[f](p) = c_1 \frac{\partial \tilde{f}}{\partial u}(u_0, v_0) + c_2 \frac{\partial \tilde{f}}{\partial v}(u_0, v_0)$$

where $\dfrac{\partial \tilde{f}}{\partial u} = \dfrac{\partial}{\partial u}(f \circ \Psi)$ and $p = \Psi(u_0, v_0)$

(iii) $v[f](p)$ is independent of the choice of parametrization Ψ.

(iv) For any C^∞ extension $\hat{f}$ of f to an open set of $\mathbb{R}^3$ containing p,

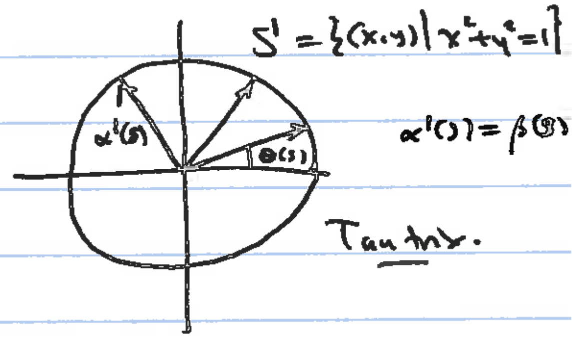
method III ⟵ $$v[f](p) = \operatorname{grad} \hat{f} \cdot v$$

Gauss Map & Shape Operator

Recall $\alpha : I \longrightarrow \mathbb{R}^2, \quad |\alpha'(s)| \equiv 1$

$$\alpha'(s) = (\cos\theta(s), \sin\theta(s)) = T(s) = \beta(s)$$

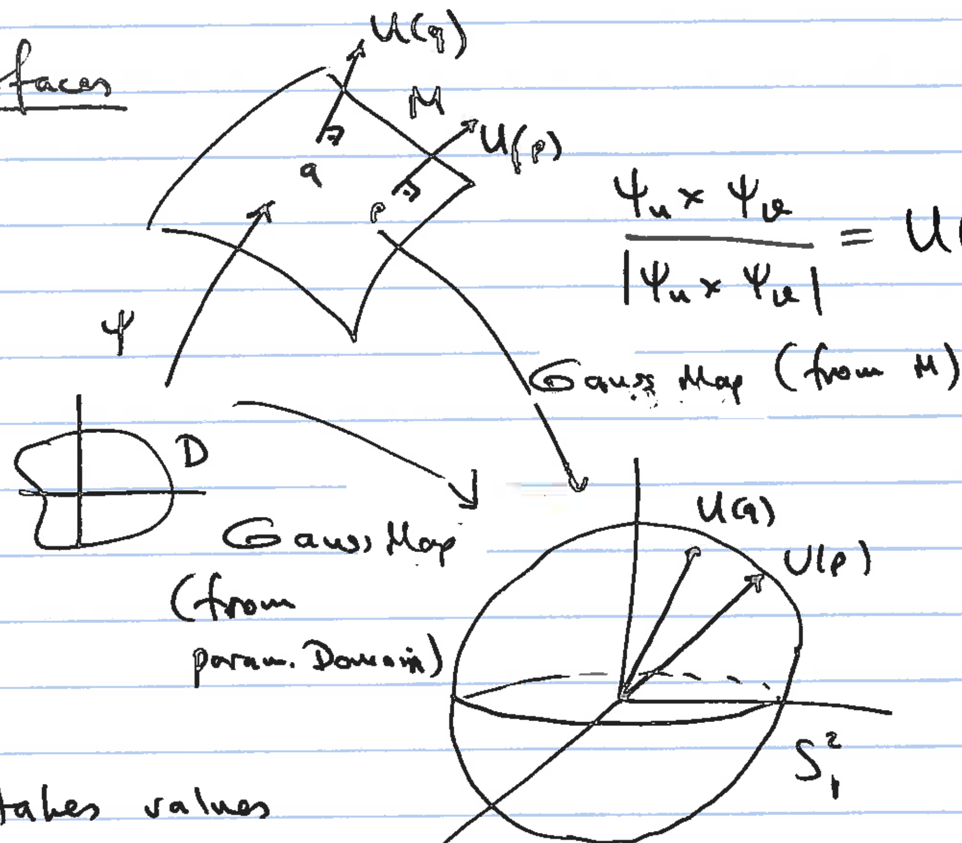
$S^1 = \{(x,y) \mid x^2+y^2=1\}$



Tantrix takes values on the unit circle.

$$\left|\frac{d\theta}{ds}\right| = \kappa_\alpha(s)$$

For surfaces



$$\frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|} = U(u,v)$$

Gauss Map takes values on the unit sphere centered at origin.

$$S^2_1 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$$

One can take the Gauss Map

$$M \longrightarrow S_1^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$$
$$p \longmapsto U(p)$$

or one can take it from the parametrization domain

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[rr, bend right] & M \arrow[r] & S_1^2 \end{tikzcd}$$

$$U(u,v) = \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}(u,v)$$

<u>Defn</u> Let U be the unit normal to a patch of a regular surface in a neighborhood of p, $v \in T_pM$

One defines

$$S_p(v) = -\nabla_v^{\mathbb{R}^3} U, \quad \text{called}$$

- the shape operator, or
- Weingarten Map.

$\nabla^{\mathbb{R}^3}$ is the covariant deriv in $\mathbb{R}^3$.

$$\nabla_v(W_1, W_2, W_3) =$$
$$= (v[W_1], v[W_2], v[W_3])$$