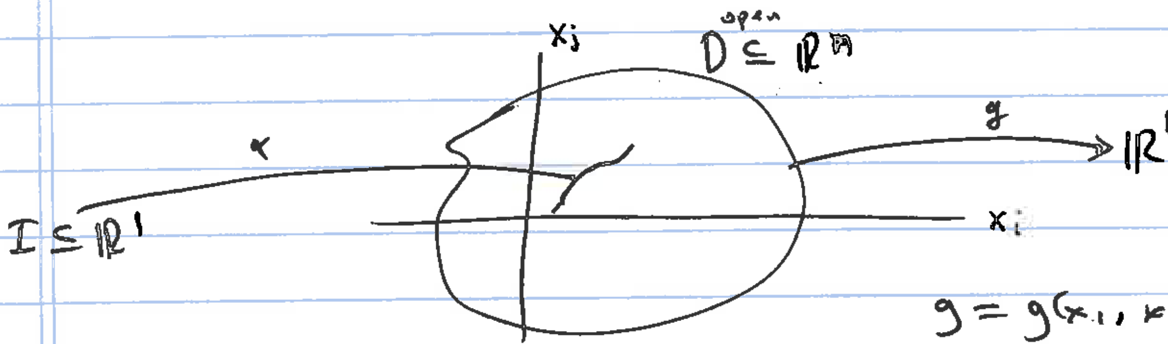
Oct 4, 2017

Differentiation in $\mathbb{R}^n$
Tangent vectors



$g = g(x_1, x_2, \dots x_n)$

$\alpha(t) = (\alpha_1(t), \dots \alpha_n(t))$

$$\frac{d}{dt}\, g(\alpha(t))(t_0) =$$

$$= \frac{\partial g}{\partial x_1}(\alpha(t_0)) \cdot \frac{dx_1}{dt}(t_0) + \frac{\partial g}{\partial x_2}(\alpha(t_0)) \cdot \frac{dx_2}{dt}(t_0) + \dots$$

$$\dots + \frac{\partial g}{\partial x_n}(\alpha(t_0)) \cdot \frac{dx_n}{dt}(t_0)$$

$$= \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\alpha(t_0)) \cdot \frac{dx_i}{dt}(t_0)$$

$$= (\operatorname{grad} g)(\alpha(t_0)) \cdot \alpha'(t_0)$$ where

$$\operatorname{grad} g = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots \frac{\partial g}{\partial x_n} \right)$$

$$\alpha' = (\alpha_1', \alpha_2', \dots \alpha_n')$$

Remark: $\forall$ curve α s.t. $\alpha(t_0) = p$, $\alpha'(t_0) = v$

we will get the same answer for

$$\left.\frac{d}{dt}\, g(\alpha(t))\right|_{t=t_0} = (\operatorname{grad} g)(\underbrace{\alpha(t_0)}_{p}) \cdot \vec{v} = (\operatorname{grad} g)(p) \cdot \vec{v}$$

<u>Def</u> ① Let $g : D^{open} \subseteq \mathbb{R}^n \xrightarrow{C^\infty} \mathbb{R}^1$

$p \in D, \quad v \in \mathbb{R}^n$

The directional derivative of g at p in direction v is

$$v[g](p) = \frac{d}{dt} g(\alpha(t))\Big|_{t=0} = (\text{grad } g)(p) \cdot v$$

where $\alpha(t)$ is any C^∞ curve in D s.t.

$\alpha(0) = p$

$\alpha'(0) = v$.

② $v[g] : D \xrightarrow{C^\infty} \mathbb{R}$ is defined pointwise by ①

a smooth function

③ Let $W : D^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be smooth vector field. $p \in D$, $v \in \mathbb{R}^n$, we define

the <u>Euclidean</u> Covariant derivative

$$(\nabla_v W)(p) = \left(v[W_1], v[W_2], v[W_3], \ldots v[W_n] \right)$$

where $W = (W_1, W_2, \ldots W_n)$

$$= \sum_{i=1}^n v[W_i] \cdot e_i.$$

$W : D^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^n \quad C^\infty$ vector field

$\Rightarrow \nabla_v W : D^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field.

Example 1 $g = x^2 + y - z^3 : \mathbb{R}^3 \to \mathbb{R}^1$

$v = (-1, 3, 2)$

$p = (1, 2, 3)$. $\operatorname{grad} g = (2x, 1, -3z^2)$

- Find $v[g](p) = (\operatorname{grad} g)(1,2,3) \cdot (-1,3,2)$

$$= (2, 1, -27) \cdot (-1, 3, 2)$$
$$= -2 + 3 - 54$$
$$= -53$$

- $v[g] = (2x, 1, -3z^2) \cdot (-1, 3, 2)$

$$= -2x + 3 - 6z^2$$

Another notation.

If we write $v = (-1, 3, 2)$ as
$$V = -\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}:$$

$$V[x^2 + y - z^3] = \left(-\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}\right)(x^2 + y - z^3)$$

$$= -2x + 3 - 6z^2$$

Ex 2 Sometimes, such as $X = y\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial z}$, vector fields have coeff. varying with the point $p = (x, y, z)$

$$X[g] = X[x^3y + yz] = \left(y\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial z}\right)[x^3y + yz]$$

$$= y\frac{\partial}{\partial x}(x^3y + yz) + 3xy\frac{\partial}{\partial z}(x^3y + yz) = y 3x^2y + 3xy \cdot y$$

Ex 3

Let $F(x,y,z) = (x^2y + z,\ xz,\ yz^2)$ in $\mathbb{R}^3$

Let $v = (3,-2,1)$.

What is $\nabla_v F = ?$

$$\nabla_{e_1} F = \frac{\partial}{\partial x}(x^2y+z,\ xz,\ yz^2) = (2xy,\ z,\ 0)$$

$$\nabla_{e_2} F = \frac{\partial}{\partial y}(x^2y+z,\ xz,\ yz^2) = (x^2,\ 0,\ z^2)$$

$$\nabla_{e_3} F = \frac{\partial}{\partial z}(x^2y+z,\ xz,\ yz^2) = (1,\ x,\ 2yz).$$

$$DF = \begin{bmatrix} 2xy & x^2 & 1 \\ z & 0 & x \\ 0 & z^2 & 2yz \end{bmatrix}$$

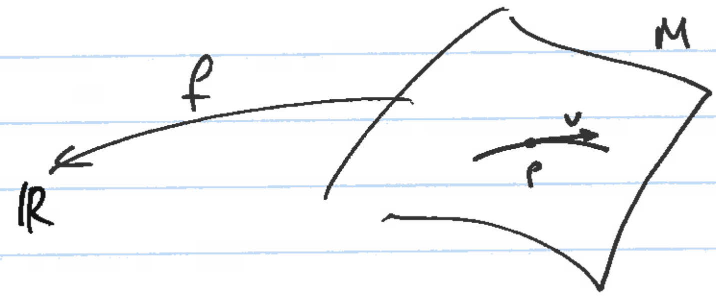
compare

$$\nabla_v F = \begin{bmatrix} 2xy & x^2 & 1 \\ z & 0 & x \\ 0 & z^2 & 2yz \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = DF \cdot v$$

$$= \begin{bmatrix} 6xy - 2x^2 + 1 \\ 3z - 2x \\ -2z^2 + 2yz \end{bmatrix}$$

Differentiation On surfaces

Defn



Let M regular surface, $p \in M$, $v \in T_pM$ be given.

Let $f: M \to \mathbb{R}$ real valued C^∞ function on M.

Define $v[f](p) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$

where α is any C^∞ curve s.t. $\alpha \subseteq M$, $\alpha(0) = p$, $\alpha'(0) = v$

Example given
- $\Psi(u,v) = (u, v, uv)$ parametrizing $z = xy$
- $f(x,y,z) = x^2 + yz$. on $M = \{(x,y,z) \mid z = xy\}$
- $p = \Psi(2,3) = (2,3,6)$
- $v = (2,4,14) \in T_pM$.

Want $v[f](p) = ?$

Need to check $v \in T_pM$, first.

$$\psi_u = (1, 0, v) \qquad \psi_u(2,3) = (1,0,3)$$
$$\psi_v = (0, 1, u) \qquad \psi_v(2,3) = (0,1,2)$$

$$(2, 4, 14) = 2(1,0,3) + 4(0,1,2)$$

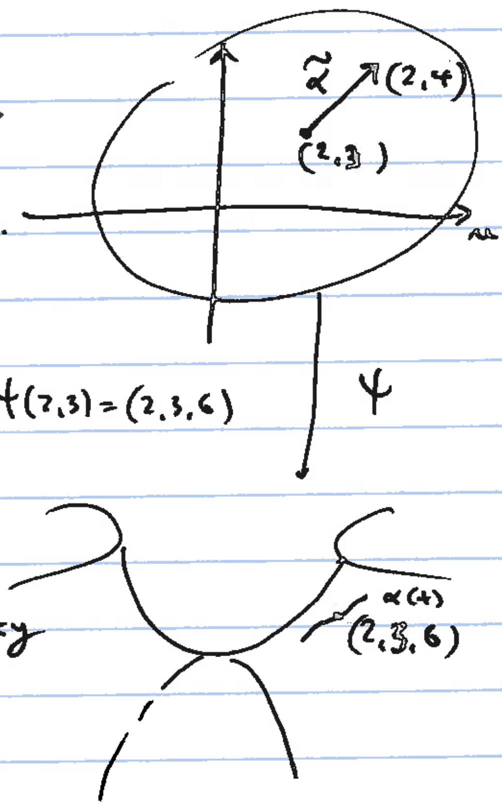
$$\Longrightarrow \quad (2,4,14) \in T_p M$$

Want $\quad v[f](p) = ?$

$$v = (2,4,14) =$$
$$= 2\psi_u(2,3) + 4\psi_v(2,3)$$

Method I

By following the definitions step by step.



$$\tilde{\alpha}(t) = (2,3) + t(2,4)$$
$$= (2+2t, 3+4t)$$
$$\tilde{\alpha}'(0) = (2,4)$$
$$\tilde{\alpha}(0) = (2,3)$$

$$\alpha(t) = \psi(\tilde{\alpha}(t))$$

$$= (2+2t, 3+4t, (2+2t)(3+4t))$$

so that $\alpha'(0) = (2\psi_u + 4\psi_v)$ at $(2,3)$

$$f(\alpha(t)) = (x^2 + yz)(\alpha(t))$$
$$= ((2+2t)^2 + (3+4t)(2+2t)(3+4t))$$

$$v[f](p) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0} = \dots = 74$$