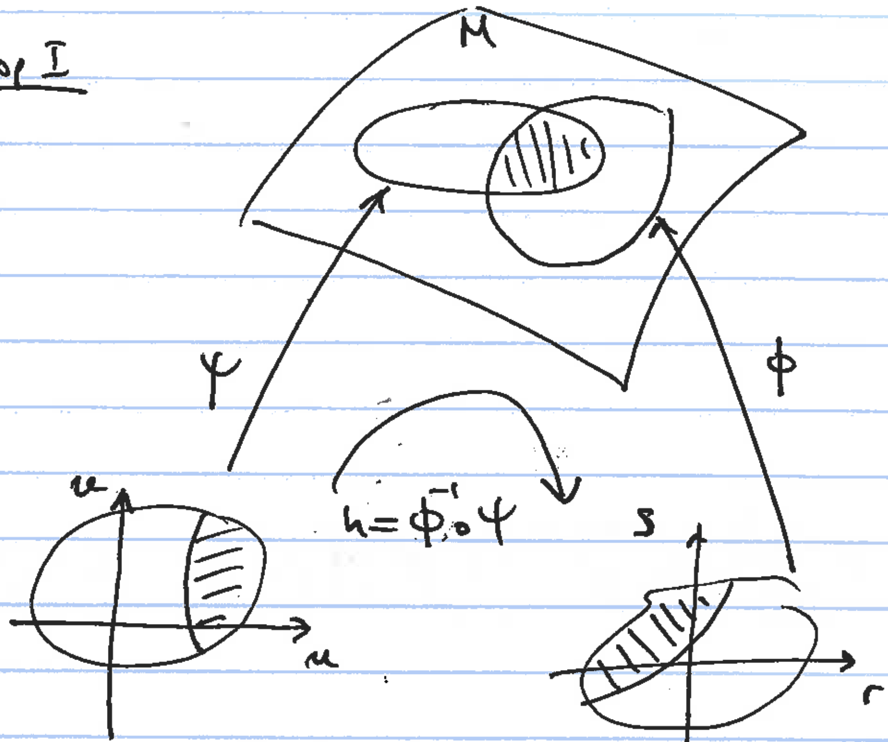
Oct, 30

①

(3.2) How do I, II, $\int f$, K change under different coordinates charts?

<u>Prop I</u>



Let ϕ, ψ be regular parametrization which overlap.

h is called the transition map

$$h(u,v) = (\phi^{-1}\circ\psi)(u,v)$$

$$(\ast)\qquad (\phi\circ h)(u,v) = \psi(u,v)$$

$$\phi = \phi(r,s) \qquad h(u,v) = (r,s)$$

$$\psi_u = \phi_r \cdot r_u + \phi_s \cdot s_u$$

$$\psi_v = \phi_r \cdot r_v + \phi_s \cdot s_v$$

$$Dh = \begin{bmatrix} r_u & r_v \\ s_u & s_v \end{bmatrix} \xrightarrow{\det}$$

$$\psi_u \times \psi_v = (\phi_r \cdot r_u + \phi_s \cdot s_u) \times (\phi_r \cdot r_v + \phi_s \cdot s_v)$$

$$= \phi_r \times \phi_s \; r_u \cdot s_v + \phi_s \times \phi_r \; s_u r_v = \phi_r \times \phi_s \,(r_u s_v - s_u r_v$$

Prop I :

i) $\Psi_u \times \Psi_v = \phi_r \times \phi_s \,(\det Dh)$

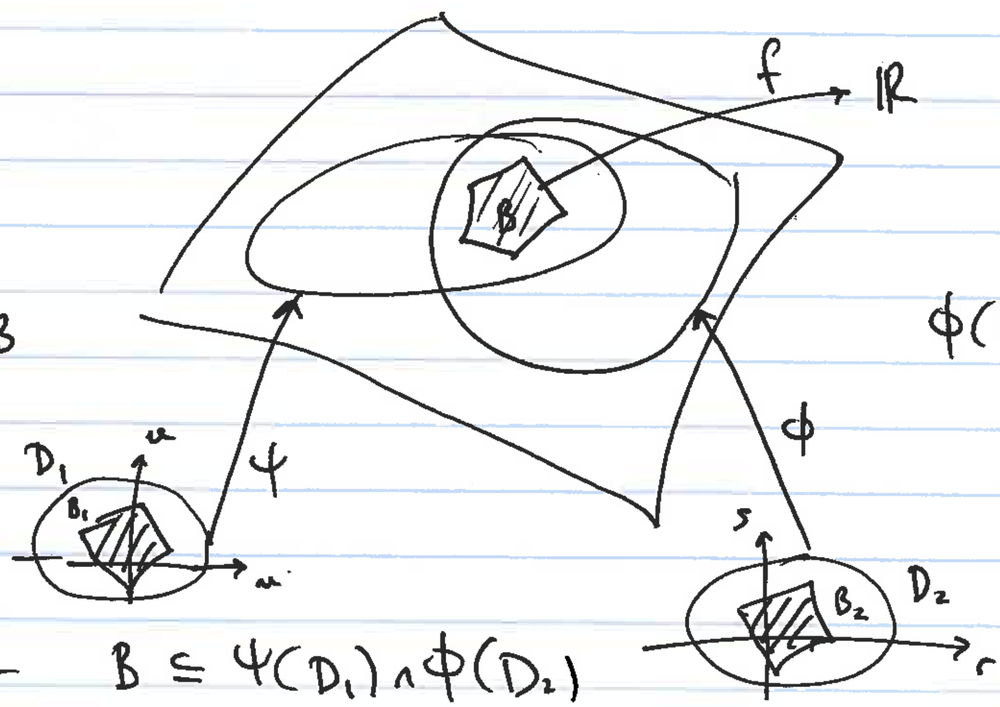
ii) $\| \Psi_u \times \Psi_v \| = \| \phi_r \times \phi_s \| \, | \mathrm{Det}\, Dh |$ ⊛⊛

iii) $U^{\Psi} = \pm U^{\phi}$

sign of det Dh.

Corollary The integration of real valued functions over M is independent of choice of parametrization

$\Psi(B_1) = B$



$\phi(B_2) = B$

Let $B \subseteq \Psi(D_1) \cap \phi(D_2)$

$$\iint_B f\, dS \underset{\text{via } \Psi}{:=} \iint_{\Psi^{-1}(B) = B_1} f(\Psi(u,v)) \, \| \Psi_u \times \Psi_v \| \, du\,dv$$

by ⊛, ⊛⊛

$$= \iint_{B_1} f(\phi \circ \underbrace{h(u,v)}_{(r,s)}) \, \| \phi_r \times \phi_s \| \, \underbrace{|\mathrm{Det}\, Dh| \; du\,dv}_{dr\,ds} = \cdots \quad \text{PTO}$$

By Change of Variables for Double integrals

integral from previous page $= \iint_{B_2} f(\phi(r,s))\,\|\phi_r \times \phi_s\|\,dr\,ds. = \iint_{B \text{ via } \phi} f\,dS$

Part II First & Second fundamental forms.

$$\left.\begin{aligned} \psi_u &= \phi_r \cdot r_u + \phi_s \cdot s_u \\ \psi_v &= \phi_r \cdot r_v + \phi_s \cdot s_v \end{aligned}\right\} \text{recall}$$

$$\begin{aligned} E^\psi = \psi_u \cdot \psi_u &= (\phi_r \cdot \phi_r) \cdot r_u^2 + 2(\phi_r \cdot \phi_s)\, r_u \cdot s_u + (\phi_s \cdot \phi_s)\, s_u^2 \\ &= E^\phi\, r_u^2 + 2F^\phi\, r_u s_u + G^\phi \cdot s_u^2 \end{aligned}$$

$$\begin{aligned} F^\psi = \psi_u \cdot \psi_v &= (\phi_r \cdot \phi_r) \cdot r_u \cdot r_v + (\phi_r \cdot \phi_s)(r_u s_v + r_v s_u) + \phi_s \cdot \phi_s\, s_u s_v \\ &= E^\phi\, r_u \cdot r_v + F^\phi\,(r_u s_v + r_v s_u) + G^\phi\, s_u s_v \end{aligned}$$

$$G^\psi = E^\phi \cdot r_v^2 + 2F^\phi\, r_v s_v + G^\phi\, s_v^2$$

These 3 equations are equivalent to:

$$\begin{bmatrix} r_u & s_u \\ r_v & s_v \end{bmatrix} \begin{bmatrix} E^\phi & F^\phi \\ F^\phi & G^\phi \end{bmatrix} \begin{bmatrix} r_u & r_v \\ s_u & s_v \end{bmatrix} = \begin{bmatrix} E^\psi & F^\psi \\ F^\psi & G^\psi \end{bmatrix}$$

proved 1

Prop II

1. $Dh^T\,[I^\phi]\,Dh = [I^\psi]$

2. $Dh^T\,[II^\phi]\,Dh = \pm\,[II^\psi]$ $\qquad \pm =$ sign of Dh

3. $K^\phi = K^\psi$.

#2 main idea:

$$\psi_u = \phi_r r_u + \phi_s s_u$$

$$\ell = U^{\psi} \cdot \psi_{uu}$$

<u>Case 1</u> $\det Dh > 0$ i.e $U^{\phi} = U^{\psi}$

$$\psi_{uu} = \phi_{rr}\, r_u \cdot r_u + \phi_{rs}\, r_s \cdot r_u + \overset{\text{tangent}}{\phi_r} \cdot r_{uu} + \phi_{sr}\, r_u \cdot s_u + \phi_{ss}\, s_u \cdot s_u + \phi_s \cdot s_{uu}$$

$$U^{\psi} \xleftrightarrow[\text{when } \det Dh > 0]{=} U^{\phi} \quad U^{\phi} \quad U^{\phi}_{\text{normal}} \quad U^{\phi} \quad U^{\phi} \quad U^{\phi}$$

$$\ell^{\psi} = \ell^{\phi} (r_u)^2 + m^{\phi} r_s r_u + 0 + m^{\phi} r_u s_u + n^{\phi} s_u s_u + 0$$

$$\begin{bmatrix} \ell^{\psi} & m^{\psi} \\ m^{\psi} & n^{\psi} \end{bmatrix} = \pm \begin{bmatrix} r_u & s_u \\ r_v & s_v \end{bmatrix} \begin{bmatrix} \ell^{\phi} & m^{\phi} \\ m^{\phi} & n^{\phi} \end{bmatrix} \begin{bmatrix} r_u & r_v \\ s_u & s_v \end{bmatrix}$$

$+$ when $U^{\phi} = U^{\psi}$ when $\det dh > 0$

$-$ when $U^{\phi} = -U^{\psi}$ when $\det dh < 0$

③ $K^{\phi} = K^{\psi}$

Recall $K = \dfrac{\ell n - m^2}{EG - F^2}$ $\left(\Longleftarrow \quad [S_p] = [I_p]^{-1} [II_p]. \quad K = \det S_p. \right)$

Prop II:

① $\Rightarrow$ $\det Dh^T \cdot \det [I^{\phi}] \cdot \det Dh = \det [I^{\psi}]$

$$(\det Dh)^2 \cdot (E^{\phi} G^{\phi} - (F^{\phi})^2) = E^{\psi} G^{\psi} - (F^{\psi})^2.$$

Recall $\det A^T = \det A$.

⑤ $\det(-A) = (-1)^n \det A$. A is $n \times n$.

Prop II ② $\Longrightarrow (\det Dh)^2 \cdot (\ell^\phi n^\phi - (m^\phi)^2) = (\pm)^2 \cdot (\ell^\psi n^\psi - (m^\psi)^2)$

$$K^\psi = \frac{\ell^\psi n^\psi - (m^\psi)^2}{E^\psi G^\psi - (F^\psi)^2} = + \frac{\ell^\phi n^\phi - (m^\phi)^2}{E^\phi G^\phi - (F^\phi)^2} = K^\phi.$$

$(\det Dh)^2$s cancel.