Oct 27, 2017

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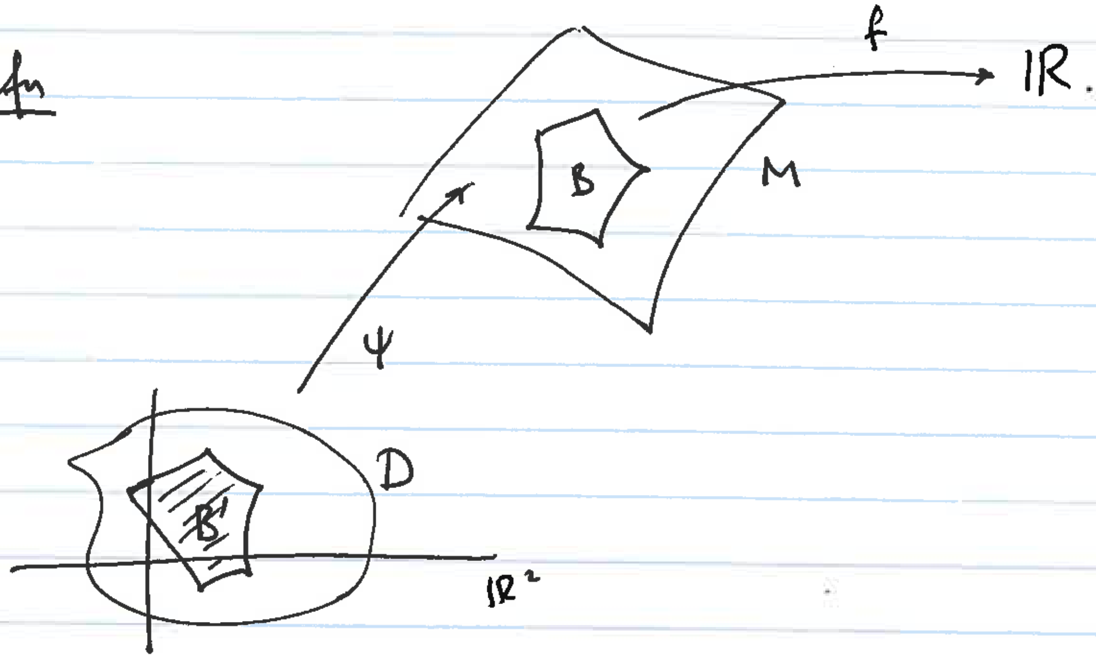
3.1 Continue

Defn



Let $\Psi : D \to M$ be a regular parametrization

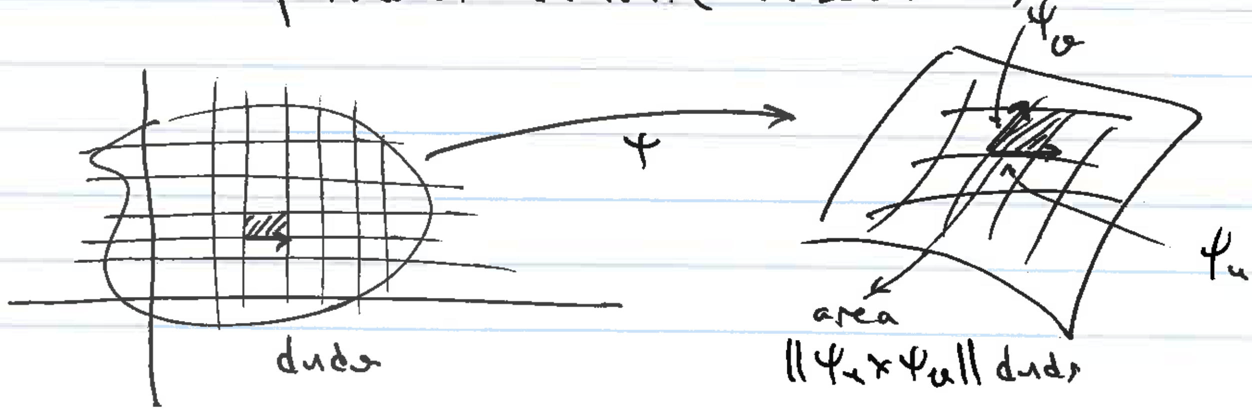
$B' \subseteq D, \quad B = \Psi(B')$

$f : M \to \mathbb{R}$.

$$\int_B f \, dS := \iint_{B'} f(\Psi(u,v)) \, \|\Psi_u \times \Psi_v\| \, du\,dv$$

Whenever RHS exist, we say LHS exists

Remark: This defn is independent of choice of parametrization. (will see in 3.2)



Corollary: $\Psi : D \to \Psi(D) \subseteq M$

$B \subseteq \Psi(D)$

$$\text{Area}(B) = \iint_B 1 \, dS = \iint_{\Psi^{-1}(B)} \|\Psi_u \times \Psi_v\| \, du\,dv$$

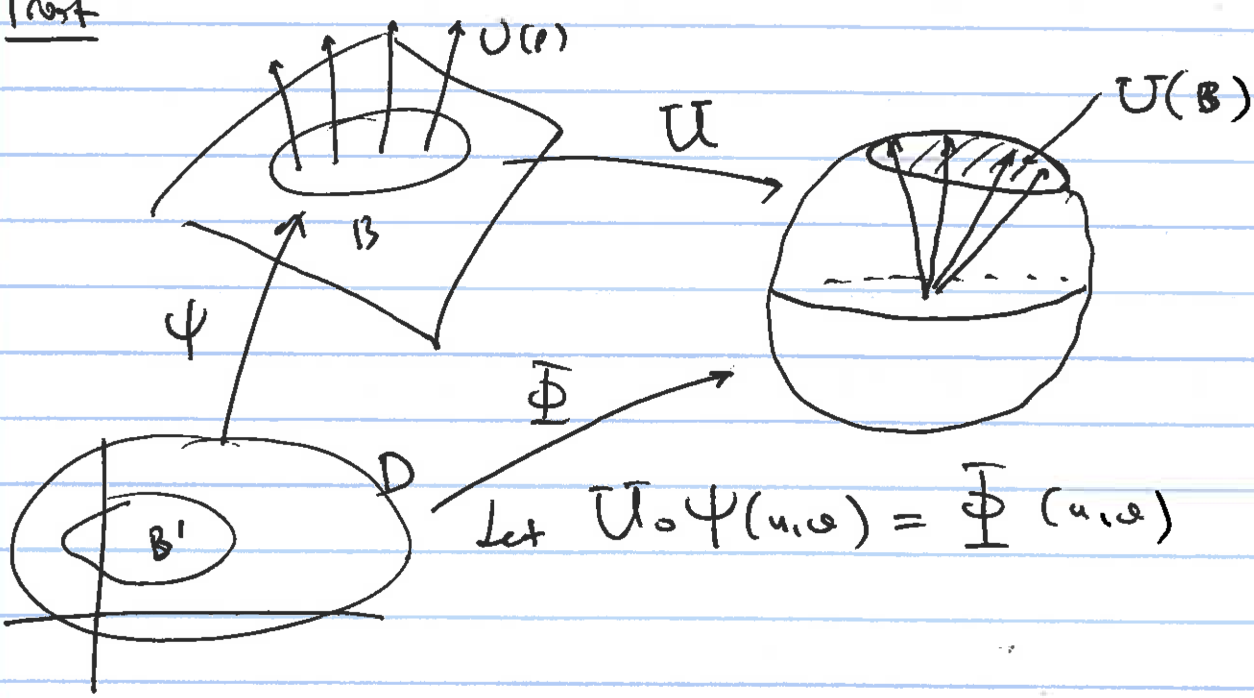
<u>Prop</u> Let M be a regular surface, Let $B \subseteq M$ s.t. K on B is > 0.

and U is 1-1 on B.

Then

$$\iint_B K \, dS = \text{area } \underbrace{U(B)}_{\text{image of } B \text{ under Gauss Map.}}$$

<u>Proof</u>



Let $U \circ \Psi(u,v) = \Phi(u,v)$

$$\Phi_u = \frac{d}{du}(U \circ \psi) = U_*(\psi_u) = -S_p(\psi_u)$$

$$\Phi_u \times \Phi_v = (-S_p(\psi_u)) \times (-S_p(\psi_v))$$

$$= S_p(\psi_u) \times S_p(\psi_v)$$

$$= \underbrace{K(p)}_{>0} \, (\underbrace{\psi_u \times \psi_v}_{\neq 0}) \quad \text{(Prop earlier.)}$$

since ψ is regular

$\Phi_u \times \Phi_v \neq 0 \implies \Phi$ is a regular param of S^2.

$$\underset{\text{in } S^2}{\text{Area}}\ U(B) = \iint_{U(B)} 1 \, dS = \iint_{B'} \|\Phi_u \times \Phi_v\| \, du\,dv$$

(via param Φ.)

$$= \iint_{B'} \|K(p) \cdot \psi_u \times \psi_v\| \, du\,dv$$

$$= \iint_{B'} K(p) \ \|\psi_u \times \psi_v\| \, du\,dv$$

$$= \iint_{B \subseteq M} K \, dS_M.$$

Similarly if $K<0$ on B, and
$\mathcal{U}$ is 1-1 on B

then $$\iint_B K\,dS = -\text{area } \mathcal{U}(B)$$

Ex

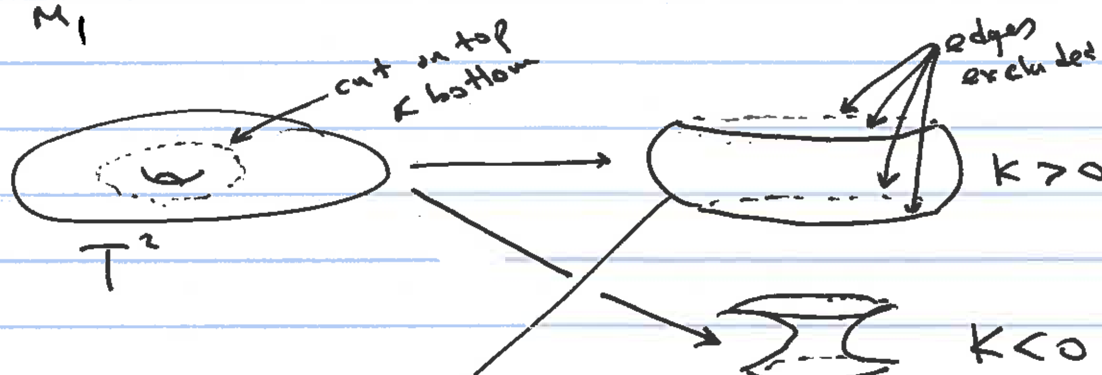
①  $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid

$$M_1 \xrightarrow{\mathcal{U}} S^2$$

$\mathcal{U}$ 1-1 & onto

$$\iint_{M_1} K\,dS = 4\pi$$

② Torus



$\mathcal{U}$ all of $S^2-\{(0,0,\pm 1)\}$

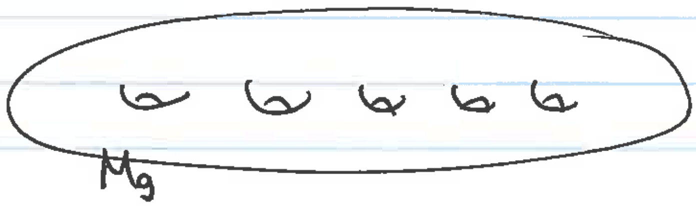
$$\iint K\,dS = 4\pi$$

$\mathcal{U}$ all of $S^2-\{(0,0,\pm 1)\}$

$$\iint K\,dS = -4\pi$$

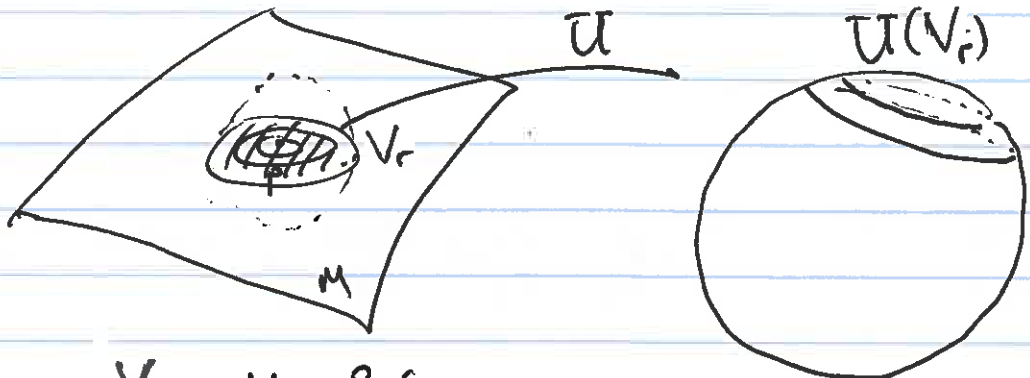
$$\iint_{T^2} K\,dS = 0$$

3 M_g $g = \#$ holes / handles $=$ genus



$$\iint_{M_g} K\,dS = 2\pi(2-2g)$$

Prop



$V_r = M \cap B_r(p)$ $r > 0$

in S^2

$$\lim_{r \to 0} \frac{\text{Area}(\mathcal{U}(V_r))}{\text{Area } V_r} = |K(p)|$$

in M

Proof: For a continuous function f at p.

$\forall \varepsilon > 0 \ \exists \delta > 0,\ |x-p| < \delta$
$\Longrightarrow |f(x) - f(p)| < \varepsilon$

i.e. $f(p) - \varepsilon \le f(x) \le f(p) + \varepsilon$

$$\forall r < \delta$$

$$\int_{V_r} \overbrace{(f(p) - \varepsilon)}^{\text{constant}} dS \leq \int_{V_r} f(x)\, dS \leq \int_{V_r} \overbrace{f(p)}^{\text{constant}} + \varepsilon$$

$$(f(p) - \varepsilon) \cdot \text{Area } V_r \leq \int_{V_r} f(x)\, dS \leq (f(p) + \varepsilon) \text{ Area } V_r$$

$$f(p) - \varepsilon \leq \underbrace{\frac{\int_{V_r} f(x)\, dS}{\text{Area } V_r}}_{\substack{\text{average of} \\ f \text{ over } V_r.}} \leq f(p) + \varepsilon$$

Take $f = |K|$ | As $\varepsilon \to 0$, $\delta \to 0$, $r \to 0$

$$|K(p)| \leq \lim_{r \to 0} \frac{\int_{V_r} |K|\, dS}{\text{Area } V_r} \leq |K(p)|. \quad (*)$$

As earlier

$$\int_{V_r} |K|\, dS = \iint_{\varphi^{-1}(V_r)} \| \Phi_u \times \Phi_v \|\, du\, dv = \text{area } \mathcal{U}(V_r) \quad (**)$$

when $K(p) > 0$ or $K(p) < 0$; & r small.
(Locally 1-1 by Inverse function Thm.)

$$(*), (**), \implies \lim_{r \to 0} \frac{\text{Area}(\mathcal{U}(V_r))}{\text{Area } V_r} = |K(p)|$$