Oct 23, 2017

①

Chap III

3.1

<u>Defn</u> Let M be a regular surface in $\mathbb{R}^3$, $p \in M$.

-) The Gauss Curvature of M at p to be

$$K(p) = \det S_p.$$ (independent of orientation) & parametrization

-) The mean curvature of M at p to be

$$H(p) = \frac{1}{2}\, \text{trace of } S_p$$ (depends on the orientation of the parametrization

<u>Remark:</u>

① Trace and determinant are characteristic values associated to a linear map.
They are independent of choices for basis.
If Φ, Ψ have the same orientation, then $S_p : T_pM \to T_pM$ satisfies

$$[S_p]_{\{\Phi_u, \Phi_v\} = \mathcal{B}_0} = P\, [S_p]_{\{\Psi_u, \Psi_v\} = \mathcal{B}_1}\, P^{-1}$$

$[S_p]_{\mathcal{B}_0}$, $[S_p]_{\mathcal{B}_1}$: They have the same trace and same determinant.

2 If ϕ and ψ have opposite orientations, then

$$-U_\psi = U_\phi$$

$$-S_p^\psi = S_p^\phi$$

$$\det[S_p^\psi]_\mathcal{B} = \det[-S_p^\psi]_\mathcal{B} = \det[S_p^\phi]_\mathcal{B} \quad \text{for any basis } \mathcal{B}.$$

$$\operatorname{tr}[-S_p^\psi] = -\operatorname{tr}[S_p^\psi].$$
$$\|$$
$$\operatorname{tr}[S_p^\phi]$$

Recall

$\det A = (-1)^n \det(-A)$ if A is $n\times n$

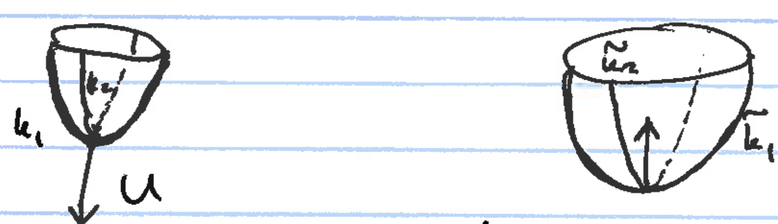
$\operatorname{tr} A = -\operatorname{tr}(-A)$

<u>Prop</u> If k_1, k_2 are principal curvatures (<u>given an orientation</u>), then

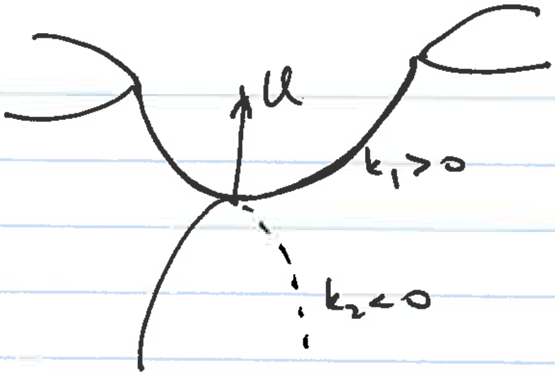
$$K(p) = k_1 k_2$$

$$H(p) = \frac{1}{2}(k_1 + k_2)$$

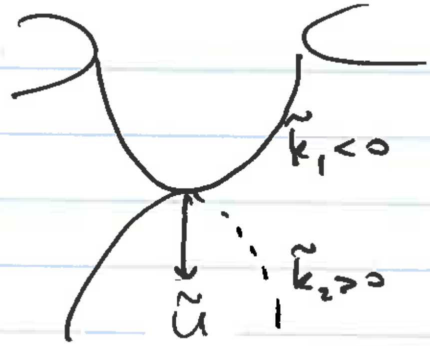
Geometric meaning of $K > 0$



$k_1 < 0$, $k_2 < 0$ $\quad k_1 k_2 > 0$ | $\tilde{k}_1 > 0$, $\tilde{k}_2 > 0$ $\quad \tilde{k}_1 \tilde{k}_2 > 0$



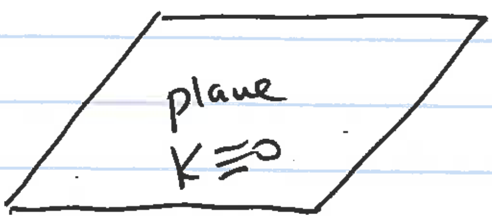
$K<0$

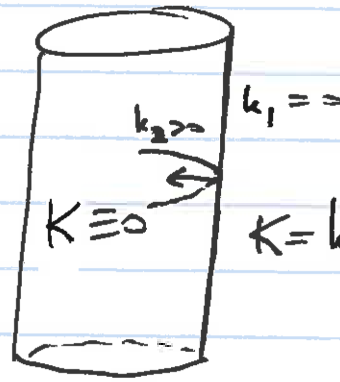


$k_1k_2<0$

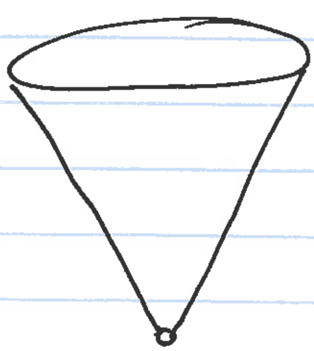
$\tilde{k}_1\tilde{k}_2<0$

$K=0$





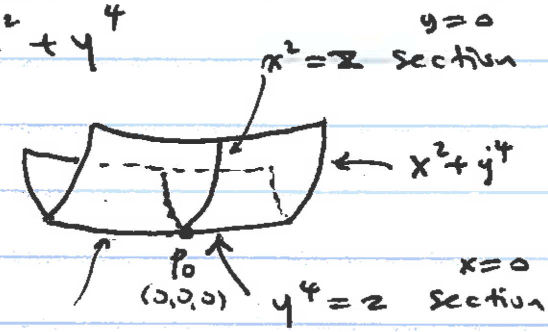
$K=k_1k_2=0$



$K\equiv 0$

Ex

$$z = x^2 + y^4$$



(x=0 section) $\quad k_2(p_0) = 0 \quad , \quad k_1(p_0) > 0 \quad$ (y=0 section)

$$K(p_0) = 0$$

$$K(p) > 0 \quad \text{if} \quad p \neq p_0$$

Prop Let $\quad \mathrm{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \qquad \mathrm{II} = \begin{bmatrix} l & m \\ m & n \end{bmatrix}.$

Then

$$K = \frac{ln - m^2}{EG - F^2}$$

$$H = \frac{Gl + En - 2Fm}{EG - F^2}$$

Proof:

$$[S_p] = [\mathrm{I}_p]^{-1}[\mathrm{II}_p] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} l & m \\ m & n \end{bmatrix}.$$

$$\det S_p = (\det \mathrm{I}_p)^{-1}(\det \mathrm{II}_p) = \frac{ln - m^2}{EG - F^2}$$

$$[S_p] = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} l & m \\ m & n \end{bmatrix}$$

$$= \frac{1}{EG-F^2}\begin{bmatrix} Gl - Fm & Gm - Fn \\ -Fl + Em & -Fm + En \end{bmatrix}$$

$$\text{trace} = \frac{Gl - Fm - Fm + En}{EG - F^2}$$

Ex 2 $\quad z = xy$, (a) Find K, H, S_p,

(b) Find principal directions & curvatures at $(0,0,0)$

(a)

$$\psi(u,v) = (u, v, uv)$$

$$\psi_u = (1, 0, v) \qquad E = 1 + v^2$$

$$\psi_v = (0, 1, u) \qquad F = uv$$

$$\psi_{uu} = (0,0,0) \qquad G = 1 + u^2$$

$$\psi_{uv} = (0,0,1) \qquad \ell = U \cdot \psi_{uu} = 0$$

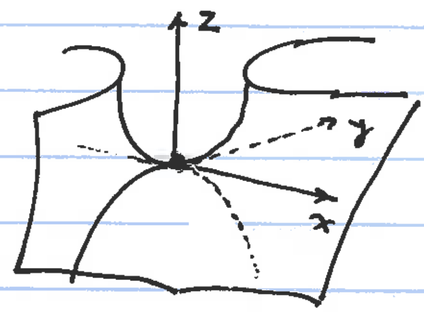
$$\psi_{vv} = (0,0,0) \qquad m = U \cdot \psi_{uv} = 1/\Delta$$

$$n = U \cdot \psi_{vv} = 0$$

$$\psi_u \times \psi_v = (-v, -u, 1)$$

$$\Delta = |\psi_u \times \psi_v| = \sqrt{1 + u^2 + v^2}$$

$$U = \frac{1}{\Delta}(-v, -u, 1)$$



$$[S_p] = [I_p]^{-1}[II_p] = \begin{bmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1/\Delta \\ 1/\Delta & 0 \end{bmatrix}$$

$$= \frac{1}{\underbrace{(1+v^2)(1+u^2) - u^2v^2}_{\Delta^2 = 1+u^2+v^2+\cancel{u^2v^2}-\cancel{u^2v^2}}} \cdot \frac{1}{\Delta} \begin{bmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[S_p]_{\{\psi_u, \psi_v\}} = \frac{1}{\Delta^3} \begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix} \qquad K = -\frac{1}{\Delta^4}$$

$$\|$$

$$K = \det \frac{1}{\Delta^3} \begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix} = \frac{1}{\Delta^6}\Big(\underbrace{u^2v^2 - (1+u^2)(1+v^2)}_{-\Delta^2}\Big)$$

$$K = -\frac{1}{(1+u^2+v^2)^2} \le 0 \quad (0 \text{ only at } (0,0,0))$$

$$H = \frac{1}{2}\,\text{trace}\,[S_p] = \frac{1}{2}\frac{1}{\Delta^3}(-2uv)$$

$$= \frac{-uv}{\Delta^3} \qquad U \uparrow \; \underset{\frac{1}{\Delta}(-v,-u,1)}{\shortparallel}$$

(b) principal curvatures / directions at $p = (0,0,0) = \psi(0,0)$

$$S_p(0,0) = \frac{1}{\Delta(0)^3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1.$$

$$\lambda^2 - 1 = 0 \qquad \lambda = \pm 1 \quad \text{eigenvalues}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b \\ -b \end{bmatrix} = \begin{bmatrix} -b \\ b \end{bmatrix} = -\begin{bmatrix} b \\ -b \end{bmatrix}.$$

$$\|\vec{u}_1\| = \|\vec{u}_2\| = 1$$

$\psi_u = (1,0,0)$ at p $\qquad \vec{u}_1 \,//\, a(1,0,0) + a(0,1,0)$

$\psi_v = (0,1,0)$ at p $\qquad \vec{u}_2 \,//\, b(1,0,0) - b(0,1,0)$

So, $\begin{cases} \vec{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) & k_1 = +1 \\ \vec{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) & k_2 = -1. \end{cases}$

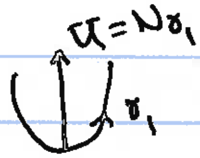
$z = xy \quad \varphi = (u, v, uv)$

$\gamma_1 = (u, u, u^2)$ a normal section in $\vec{u}_1$ direction

$\gamma_1' = (1, 1, 2u) \qquad \gamma_1'(0) = (1, 1, 0)$

$\gamma_1'' = (0, 0, 2) \qquad \gamma_1''(0) = (0, 0, 2)$

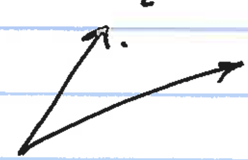
$$k_{\gamma_1}(0) = \frac{|(\gamma_1' \times \gamma_2'')(0)|}{|\gamma_1'(0)|^3}$$



$$= \frac{|(2, -2, 0)|}{|(1,1,0)|^3} = \frac{\sqrt{4+4}}{(\sqrt{1+1})^3}$$

$$= \frac{\sqrt{8}}{(\sqrt{2})^3} = 1. \quad \overset{\text{ok}}{(k_1 = +1)}$$

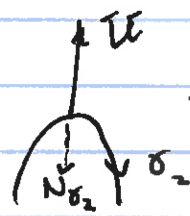
$\gamma_2 = (u, -u, -u^2) \; \dashrightarrow \; k_{\gamma_2}(0) = +1$



$(k_2 = -1)$

$\gamma_2''(0) = (0, 0, -2)$

$U(0,0) = (0, 0, 1)$



$\Longrightarrow \quad k_2 < 0$

Recall $\begin{cases} \alpha''(0) \cdot U(\alpha'(0)) = \mathrm{II}_p(\alpha'(0)) \\ k_n(\alpha'(0)) = k_\alpha(0) \cdot \cos\theta \longrightarrow \end{cases}$ $\theta = \measuredangle(U, N_\alpha)$

For γ_1, $\theta = 0$

For γ_2, $\theta = \pi$