(2.4)

More Linear Algebra:

Prop: $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a symmetric linear map,
$(v \cdot L(w) = L(v) \cdot w \ \forall v, w)$.
Let v_1, v_2 be a set of eigenvectors,
$L(v_i) = \lambda_i v_i$ with eigenvalues $\lambda_1 \geq \lambda_2$, $|v_1| = |v_2| = 1$.
Then

1) For $v = \cos\theta\, v_1 + \sin\theta\, v_2$, one has
$$v \cdot L(v) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$
2) $$\lambda_2 \leq L(v) \cdot v \leq \lambda_1, \ \forall v \in \mathbb{R}^2, \ |v| = 1.$$
3) If $v \cdot L(v) = \lambda_i$ for $|v| = 1$, then $v = \pm v_i$ for $\lambda_1 > \lambda_2$

Proof of $\lambda_1 = \lambda_2 = \lambda \implies L = \lambda\, Id$.

$$\left(P^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} P = P^{-1} \lambda I P = \lambda P^{-1} I P = \lambda I \right)$$

1) $v = \cos\theta\, v_1 + \sin\theta\, v_2$
$$v \cdot L(v) = v \cdot \lambda v = \lambda \cdot v \cdot v = \lambda, \text{ for } |v| = 1$$
2) $$\lambda \leq v \cdot L(v) \leq \lambda \quad \text{for } |v| = 1.$$

Case 2 $\lambda_1 > \lambda_2$

Obs 1. $v_1 \perp v_2$

Choose any $w \perp v_1$, $|w| = 1$.
$$L(w) = aw + b v_1$$
$$v_1 \cdot L(w) = v_1 \cdot (aw + b v_1) = 0 + b = b.$$
$$\parallel$$
$$L(v_1) \cdot w = \lambda_1 v_1 \cdot w = \lambda_1 \underbrace{v_1 \cdot w}_{0} = 0. \qquad b = 0$$

$L(w) = aw$ i.e. w is an eigenvector $\left.\begin{matrix} \\ w \perp v_1 \end{matrix}\right\} \Rightarrow v_2 \parallel w$, $v_2 \perp v_1$

Case 2

$$\lambda_1 > \lambda_2$$

$$v = v_1 \cos\theta + v_2 \sin\theta$$

$$L(v_i) = \lambda_i v_i \,, \quad v_1 \perp v_2$$

$|v_1| = |v_2|,\ v_1 \perp v_2$

$$v = v_1 \cos\theta + v_2 \sin\theta \quad (\Rightarrow |v| = 1)$$

$v_1 \perp v_2$

$|v_1| = |v_2| = 1$

$$L(v) = \lambda_1 v_1 \cos\theta + \lambda_2 v_2 \sin\theta .$$

$$v \cdot L(v) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta .$$

$$\lambda_2 \leq \lambda_2 + \underbrace{(\lambda_1 - \lambda_2)}_{>}\underbrace{\cos^2\theta}_{\geq 0} = v \cdot L(v) = \lambda_1 + \underbrace{(\lambda_2 - \lambda_1)}_{<0}\underbrace{\sin^2\theta}_{\geq 0} \leq \lambda_1$$

$=$ when $\cos\theta = 0$

$$\theta = \pm \frac{\pi}{2}$$

$v = \pm v_2$ since both unit vectors

$=$ when $\sin\theta = 0$

$$\theta = 0, \pi .$$

$$v = \pm v_1 .$$

Defn For the symmetric linear map,
Shape operator $S_p : T_pM \to T_pM$:

1) The eigenvalues are called principal curvatures $k_1 \geq k_2$

2) The corresponding eigenvectors are called principal directions $\vec{u}_1, \vec{u}_2$

$$S_p(\vec{u}_1) = k_1 \vec{u}_1$$
$$S_p(\vec{u}_2) = k_2 \vec{u}_2$$

Consequently

$$k_2 \leq \overbrace{S_p(\vec{u}) \cdot \vec{u}}^{\mathrm{II}_p(\vec{u})} \leq k_1$$

$$\underset{\min \mathrm{II}_p(u)}{\overset{\|}{k_2}} \qquad\qquad \underset{\max \mathrm{II}_p(u)}{\overset{\|}{k_1}}$$

only when $u = \pm u_2$ only when $u = \pm u_1$

If $k_1 > k_2$.

(Caution: if $k_1 = k_2$, then $\forall v, |v| = 1$ $\Rightarrow \mathrm{II}_p(v) = k_1$)

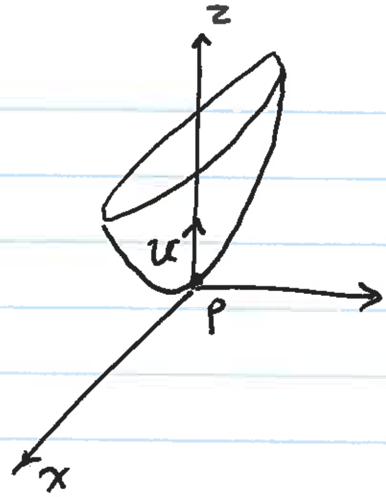
Euler's Formula

$$k_n(\underbrace{u_1 \cos\theta + u_2 \sin\theta}_{\text{unit vector } u}) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

normal curvature

$$\theta = \measuredangle(\vec{u}, \vec{u}_1)$$

Ex (a) $z = x^2 + 2y^2$



$$
\begin{array}{lll|l}
 & & & \text{at } (0,0) \\
\Psi(u,v) & = & (u, v, u^2 + 2v^2) & \\
\Psi_u & = & (1, 0, 2u) & (1,0,0) \\
\Psi_v & = & (0, 1, 4v) & (0,1,0) \\
\Psi_{uu} & = & (0, 0, 2) & (0,0,2) \\
\Psi_{uv} & = & (0, 0, 0) & (0,0,0) \\
\Psi_{vv} & = & (0, 0, 4) & (0,0,4) \\
 & & \text{at } (0,0) &
\end{array}
$$

$\Psi(0,0) = (0,0,0) = p$

$U(0,0) = \Psi_u(0,0) \times \Psi_v(0,0) / \text{length}$

$= (1,0,0) \times (0,1,0) / 1$

$U(0,0) = (0,0,1)$

at (0,0)

$E = \Psi_u \cdot \Psi_u = 1$

$F = \Psi_u \cdot \Psi_v = 0$

$G = \Psi_v \cdot \Psi_v = 1$

at (0,0)

$\ell = U \cdot \Psi_{uu} = 2$

$m = U \cdot \Psi_{uv} = 0$

$n = U \cdot \Psi_{vv} = 4.$

$$[S_p] = [I_p]^{-1}[II_p] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\text{diagonal already}}$$

wrt basis $\Psi_u = (1,0,0)$

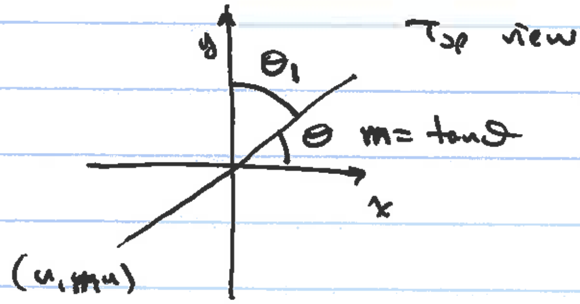
$\Psi_v = (0,1,0)$

$k_1 = 4, \quad k_2 = 2$

$\boxed{\text{eigenvector } (0,1,0)}$ $\quad$ $\boxed{\text{eigenvector } (1,0,0)}$

principal directions

Normal cross sections



$$\psi(u, mu)$$

$$\gamma_m = (u, mu, u^2 + 2m^2u^2)$$

$$\gamma_m' = (1, m, (1+2m^2)\cdot 2u)$$

$$\gamma_m'' = (0, 0, 2(1+2m^2))$$

$$\gamma_m' \times \gamma_m'' = (2m(1+2m^2), -2(1+2m^2), 0)$$

$$k_{\gamma_m}(0) = \frac{|\gamma_m' \times \gamma_m''|}{|\gamma_m'|^3} = \frac{(1+2m^2)\sqrt{4m^2+4}}{(1+m^2)^{3/2}} = \frac{2(1+2m^2)}{1+m^2}$$

$$\gamma_m'(0) = (1, m, 0)$$

$$m = \tan\theta . \qquad (\theta = \measuredangle(\underbrace{x\text{-axis}}_{u_2 \text{ direction}}, \vec{u}))$$

$$k_{\gamma_m}(0) = 2\left(1 + \frac{m^2}{1+m^2}\right) = 2\left(1 + \frac{\tan^2\theta}{\sec^2\theta}\right)$$

$$= 2(1 + \sin^2\theta)$$

$$= 2\cos^2\theta + 4\sin^2\theta$$

Caution $\theta = \measuredangle(\vec{u}_2, \vec{u})$

In Euler's formula $\theta_1 = \measuredangle(\vec{u}_1, \vec{u})$

In the 1st quadrant: $\theta + \theta_1 = \frac{\pi}{2}$.