Oct 18, 2017

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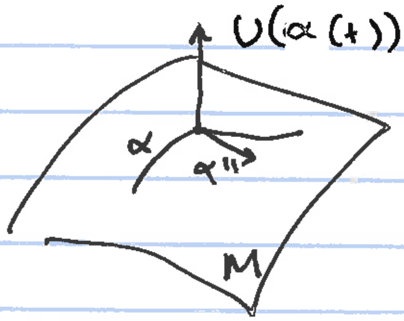
2.4 Normal Curvature

Prop Let $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ be a C^∞ curve.
Then

$$\alpha''(t) \cdot U(\alpha(t)) = S_{\alpha(t)}(\alpha'(t)) \cdot \alpha'(t)$$

$$= \mathrm{II}_{\alpha(t)}(\alpha'(t))$$



Proof $\underbrace{\alpha'(t)}_{\text{tangential}} \perp \underbrace{U(\alpha(t))}_{\text{normal.}}$

$$\alpha'(t) \cdot U(\alpha(t)) = 0$$

$$0 = \frac{d}{dt}\left(\alpha'(t) \cdot U(\alpha(t))\right) = \alpha''(t) \cdot U(\alpha(t)) + \alpha'(t) \cdot \frac{d}{dt} U(\alpha(t))$$

$$= \alpha''(t) \cdot U(\alpha(t)) + \alpha'(t) \cdot \nabla_{\alpha'(t)} U$$

$$= \alpha''(t) \cdot U(\alpha(t)) + \alpha'(t) \cdot \left(-S_{\alpha(t)}(\alpha'(t))\right)$$

$$0 = \alpha''(t) \cdot U(\alpha(t)) - \alpha'(t) \cdot S_{\alpha(t)}(\alpha'(t)).$$

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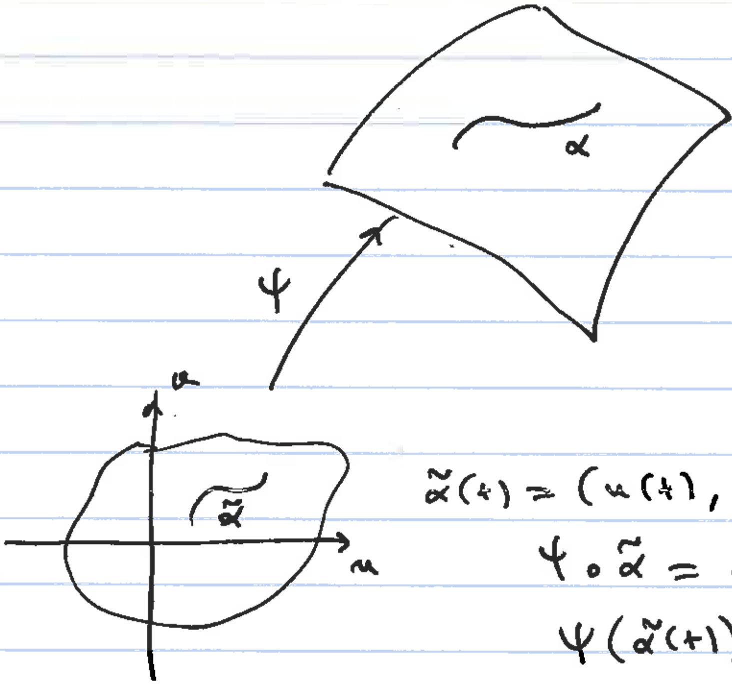
Same calculation in local coordinates



$$\tilde{\alpha}(t) = (u(t), v(t))$$

$$\psi \circ \tilde{\alpha} = \alpha.$$

$$\psi(\tilde{\alpha}(t)) = \psi(u(t), v(t)) = \alpha(t)$$

$$\alpha' = \psi_u \cdot u' + \psi_v v'$$

$$\alpha'' = \psi_{uu} u' \cdot u' + \psi_{uv} v' \cdot u' + \psi_u \cdot u'' + $$
$$+ \psi_{vu} u' \cdot v' + \psi_{vv} v' \cdot v' + \psi_v \cdot v''$$

$$(\alpha'' \cdot U) = (U \cdot \psi_{uu})(u')^2 + (U \cdot \psi_{uv}) v' u' + 0 +$$
$$+ (U \cdot \psi_{vu}) u' v' + (U \cdot \psi_{vv}) v' \cdot v' + 0$$

$$[\mathrm{II}_p] = \begin{bmatrix} l & m \\ m & n \end{bmatrix}$$

$$= l(u')^2 + m v' u' + m u' v' + n (v')^2$$

$$= [u', v'] \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \mathrm{II}_{\alpha(t)}(\alpha'(t))$$

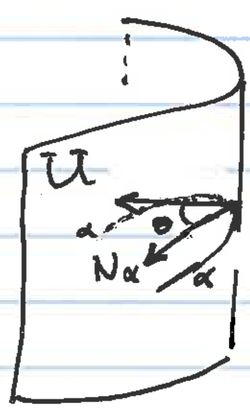
$$= \alpha'(t) \cdot S_{\alpha(t)}(\alpha'(t))$$

<u>Defn</u> Let M be a regular surface, $p \in M$, $\vec{w} \in T_pM$, $|w| = 1$. Define the <u>normal curvature</u> of M at p in the $\vec{w}$ direction to be

$$\underset{\text{I will use}}{k_n}(w) = \underset{\text{book}}{k}(w) = S_p(w) \cdot w = \mathrm{II}_p(w)$$

<u>Caution</u>	$k = k_n$	normal curvature
Greek Letter (koppa)	κ	curve curvature in $\mathbb{R}^3$
	K	Gaussian curvature

<u>Prop.</u> Let $\alpha : (-\varepsilon, \varepsilon) \xrightarrow{C^\infty} M \subseteq \mathbb{R}^3$, M reg. surf. $|\alpha'| = 1$,



Let κ_α, N_α be the curve curvature, and curve principal normal, as α is considered in $\mathbb{R}^3$.

Let $\theta = \angle(N_\alpha(0), U(\alpha(0)))$ then

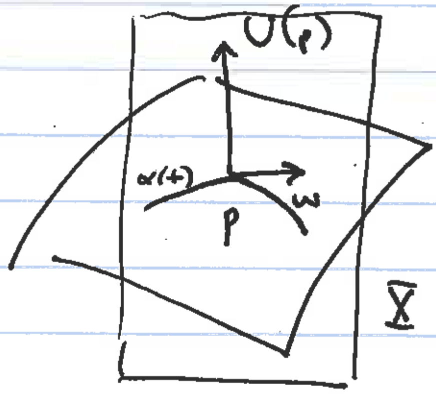
$$k_n(\alpha'(0)) = \kappa_\alpha(0) \cdot \cos\theta$$

<u>Proof</u>

$$\begin{aligned} k_n(\alpha'(0)) &= S_{\alpha(0)}(\alpha'(0)) \cdot \alpha'(0) && \text{defn.} \\ &= \alpha''(0) \cdot U(\alpha(0)) && (1^{st}\text{ prop.})\ p.1 \\ &= \overbrace{\kappa_\alpha(0) \cdot N_\alpha(0)} \cdot U(\alpha(0)) \\ &= \kappa_\alpha(0) \cdot \cos\theta \, \underbrace{|N_\alpha(0)|}_{1} \cdot \underbrace{|U(\alpha(0))|}_{1} \end{aligned}$$

Q: $\exists$ any α s.t. $\kappa_\alpha(0) = k_n(\alpha(0))$?

Normal Sections



Let $w \in T_pM$

Π is a plane thru p, $/\!/ \ w$ and $/\!/ \ U(p)$.

Let α be the curve of intersection
α is called a normal section.

Implicit Function Thm $\Longrightarrow$ α is a diff'ble curve

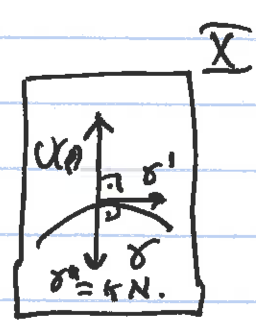
__Prop__ Let M be a regular surface, $p \in M$, $w \in T_pM$, and $|w| = 1$. Let γ be the curve of intersection of M with the plane passing thru p, parallel to w and $U(p)$, $|\gamma'| \equiv 1$

Then
$$k_n(\gamma'(0)) = \pm \kappa_\gamma(0)$$

__Proof__ since $\gamma \subseteq$ plane Π

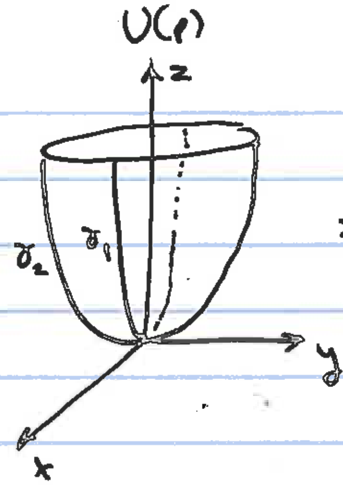
$$\gamma', \gamma'', N_\gamma \ /\!/ \ \Pi$$
$$\gamma' = w$$
$$\kappa N_\gamma = \gamma'' \perp \gamma' = w$$
$$N_\gamma /\!/ \ U(p)$$

Remark:
If $\gamma''(0) = 0$, then
$\kappa_\gamma(0) = 0 = k_n(\gamma'(0))$
automatically, from def^{n}s.



$$\Theta = \measuredangle(N_\gamma(0), U(\gamma(0))) = \pi \text{ or } 0$$
$$k_n(\gamma'(0)) = \pm \kappa_\gamma(0).$$

Ex (A)



$z = 4x^2 + y^2$

Σ_1 plane thru O, containing x & z axes

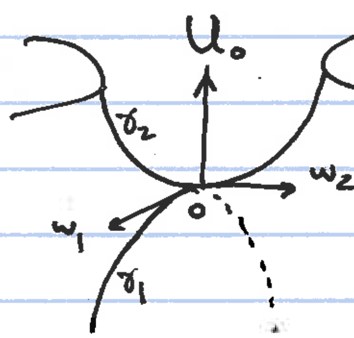
Σ_2 plane thru O, containing y & z axes

$\gamma_1 = M \cap \Sigma_1$
$\gamma_2 = M \cap \Sigma_2$

$U(p) \uparrow$

$$\left.\begin{aligned} \kappa_{\gamma_1}(0) &= \mathrm{II}_{\vec{0}}((1,0,0)) \\ \kappa_{\gamma_2}(0) &= \mathrm{II}_{\vec{0}}((0,1,0)) \end{aligned}\right\} \quad \kappa_{\gamma_1}(0) > \kappa_{\gamma_2}(0) > 0$$

(B)



$z = xy$ surface

$\mathrm{II}_0(w_2) > 0$

$\mathrm{II}_0(w_1) < 0.$

	HW.	MT
A -	≥ 90	80
B -	≥ 80	65
C -	≥ 65	50
D -	≥ 50	35

median 72