Oct 16, 2017

1

2.3 to finish

Ex $\phi(u,v) = (u, v, 2uv)$

$\phi_u = (1, 0, 2v)$ $\quad E = \phi_u \cdot \phi_u = 1 + 4v^2$

$\phi_v = (0, 1, 2u)$ $\quad F = \phi_u \cdot \phi_v = 4uv$

$\quad G = \phi_v \cdot \phi_v = 1 + 4u^2$.

$\phi_{uu} = (0, 0, 0)$ $\quad l = \phi_{uu} \cdot U = 0$

$\phi_{uv} = (0, 0, 2)$ $\quad m = \phi_{uv} U = \frac{2}{\Delta}$

$\phi_{vu} = (0, 0, 2)$

$\phi_{vv} = (0, 0, 0)$ $\quad n = \phi_{vv} \cdot U = 0$

$\phi_u \times \phi_v = (-2v, -2u, 1)$

$\Delta = |\phi_u \times \phi_v| = \sqrt{4v^2 + 4u^2 + 1}$.

$U = \frac{1}{\Delta}(-2v, -2u, 1)$

$$[S_p] = [I_p]^{-1}[II_p] = \begin{bmatrix} 1+4v^2 & 4uv \\ 4uv & 1+4u^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{2}{\Delta} \\ \frac{2}{\Delta} & 0 \end{bmatrix}.$$

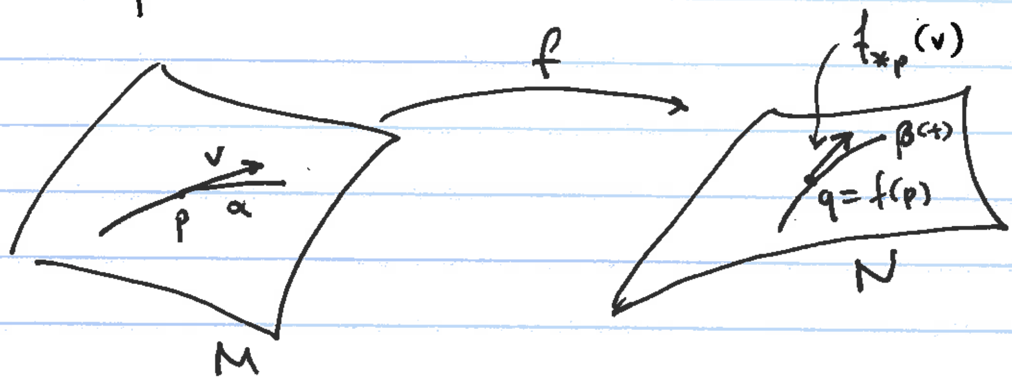
$$= \frac{1}{(1+4u^2)(1+4v^2) - 16u^2v^2} \begin{bmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\Delta} \\ \frac{2}{\Delta} & 0 \end{bmatrix}$$

$$= \frac{1}{1 + 4u^2 + 4v^2} \begin{bmatrix} \frac{-8uv}{\Delta} & \frac{2+8u^2}{\Delta} \\ \frac{2+8v^2}{\Delta} & \frac{-8uv}{\Delta} \end{bmatrix}$$

$$= \frac{2}{\Delta^3} \begin{bmatrix} -4uv & 1+4u^2 \\ 1+4v^2 & -4uv \end{bmatrix}.$$

Def

Let $f: M \to N$ be a diff'ble map where M, N are regular surfaces.
Let $p \in M$.



The derivative $f_{*p} : T_pM \longrightarrow T_{f(p)}N = T_qN$

is defined as follows:

Let $v \in T_pM$, choose any curve $\alpha(t): (-\varepsilon, \varepsilon) \to M$
s.t. $\alpha(0) = p$.
$\alpha'(0) = v$.

Then let $\beta(t) = f(\alpha(t))$

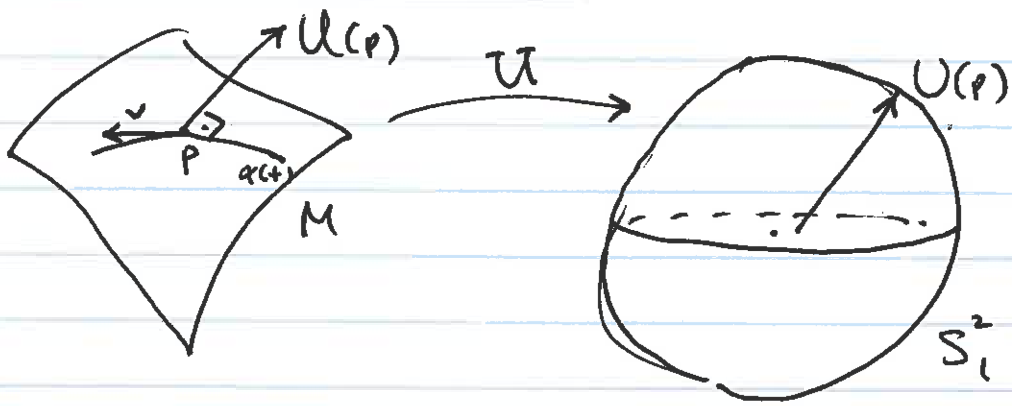
$\beta(0) = f(\alpha(0)) = f(p) = q$.

$\beta'(0) = f_{*p}(v)$.

Other Notation $f_* = Df$ is commonly used, for the derivative function.

Important Example

$$U : M \longrightarrow S^2_1 \qquad \text{Gauss Map.}$$
$$p \longmapsto U(p)$$



$= \{(x,y,z) \mid x^2+y^2+z^2=1\}$

Let $v \in T_pM$; What is $U_{*p}(v) = ?$ choose $\alpha(t)$ in M s.t.

$\alpha(0) = p$

$\alpha'(0) = v.$

$$\beta(t) = U(\alpha(t))$$
$$\beta(0) = U(\alpha(0)) = U(p)$$

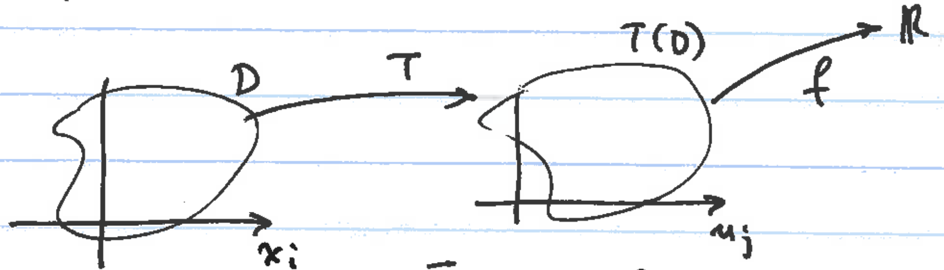
$$\beta'(0) = \beta'(t)\Big|_{t=0} = \frac{d}{dt} U(\alpha(t))\Big|_{t=0} = \nabla_{\alpha'(0)} U$$
$$= \nabla_v U$$
$$= -S_p(v).$$

Recall $T_{U(p)}S^2 \perp U(p)$
$T_pM \perp U(p)$

$$(DU_p(v) =) \quad U_{*p}(v) = -S_p(v) \in T_{U(p)}S^2 \stackrel{\text{"}}{=} T_pM$$

Derivative of U (Gauss Map) is the shape operator.

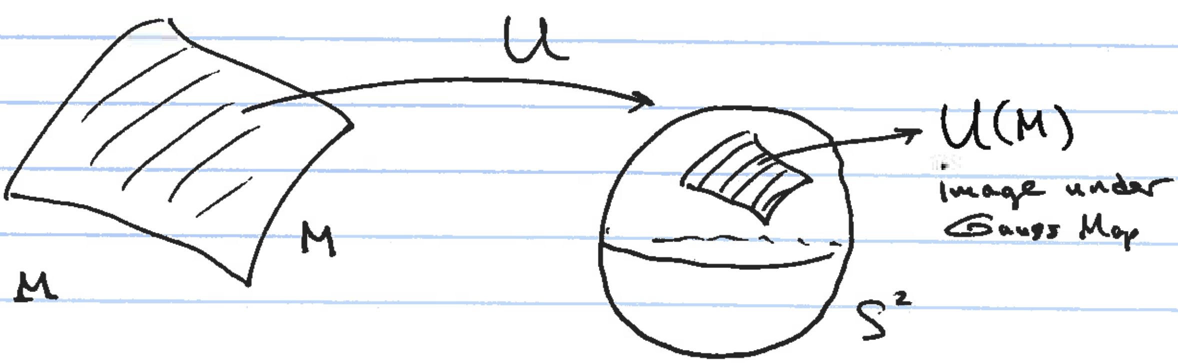
Recall (Jacobi's Thm)



$\vec{u} = T(\vec{x})$

$$\int_{T(D)} f(\vec{u})\, du_1 d_2 \cdots du_n =$$

$$= \int_D f(T(\vec{x})) \cdot |\det DT|\, dx_1 \cdots dx_n$$



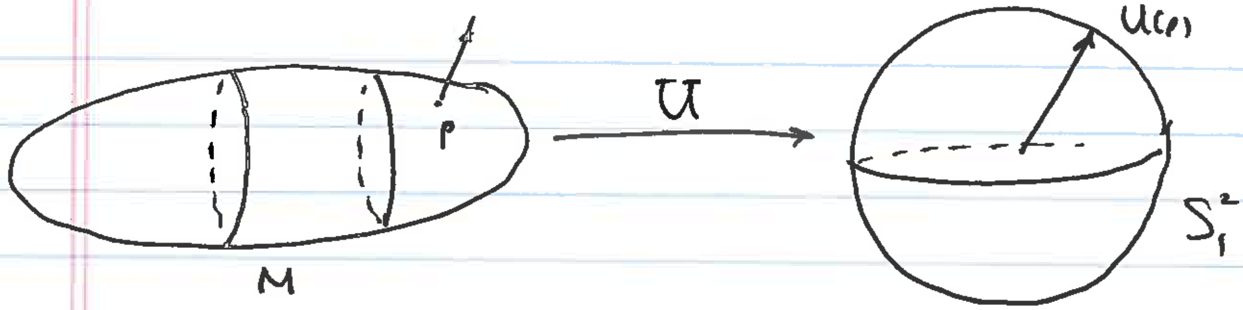
Assume
U is 1-1 on M
for simplicity

$$\iint_{U(M)} 1 \cdot dS_{S^2} = \iint_M 1 \cdot |\det DU|\, dS_M$$

$$DU_p = U_{*p} = \text{derivative of } \underset{\text{at } p}{U} = -S_p .$$

$$\det S_p = K \quad \text{Gaussian Curvature}$$

$$\iint_M |K|\, dS_M = \iint_{U(M)} 1 \cdot dS_{S^2} = \text{area } U(M)$$



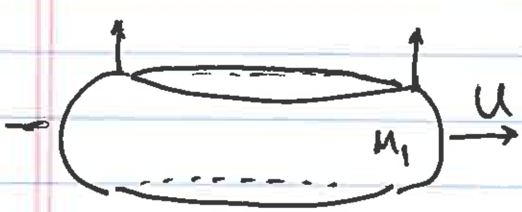
Convex ovaloid $\qquad$ U is 1-1 and onto S^2_1.

$$\int_M K\, dS_M = \int 1 \cdot dS_{S^2} = \text{area}(S^2_1) = 4\pi$$

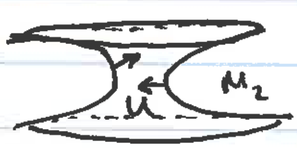
Gauss Bonnet Later:
(Last day of classes)

$$\iint_M K\, dS = 4\pi$$

Torus cut into two pieces



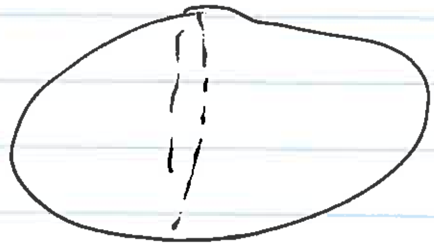
$U(M_1) =$ all of S^2 covered with $+$ orientation



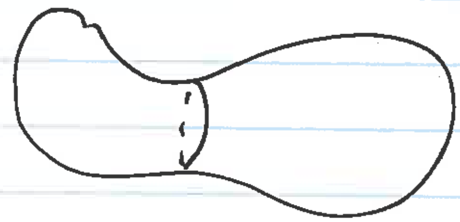
$U(M_2) =$ all of S^2 covered with $-$ orientation

$$\int_M K\, dS = \int_{M_1} K\, dS + \int_{M_2} K\, dS = 4\pi - 4\pi = 0.$$

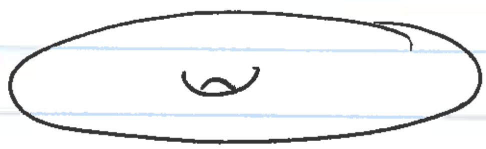
Informally $g = \#$ holes $= \#$ handles



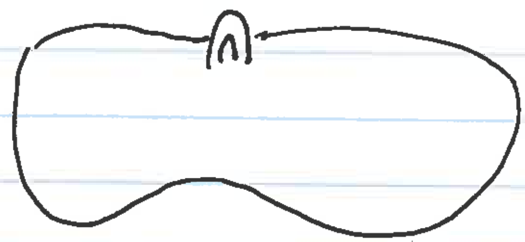
$g=0$

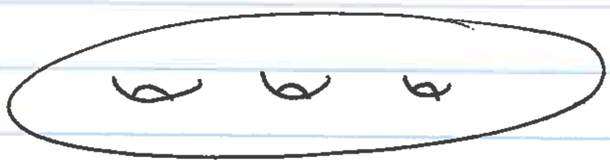


$g=1$

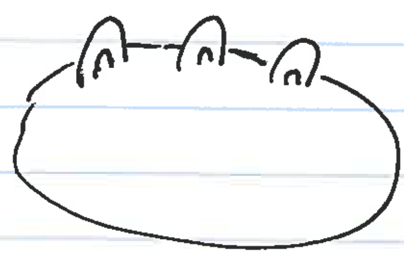


$g=1$





$g=3$



Gauss Bonnet Thm says

$$\int_{M_g} K\,ds = 2\pi\,(2-2g)$$

as long as M_g is C^∞, regular, without boundary, compact, oriented genus g 2-surface.