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①

Prop $S_p : T_pM \to T_pM$ is a symmetric linear operator and $T_pM \cong \mathbb{R}^2$

Conclusion/Defn

- S_p has two real eigenvalues λ_1, λ_2
- $\exists \{v_1, v_2\}$ an orthonormal basis for T_pM s.t.

$$S_p(v_1) = \lambda_1 v_1$$
$$S_p(v_2) = \lambda_2 v_2$$

- λ_1, λ_2 are called principal curvatures
- v_1, v_2 are called principal directions.

Defn Let M be a regular surface, $p \in M$. Let $\Psi : D \subseteq \mathbb{R}^2 \to M$ be a regular parametrization around p, so that

$$\mathcal{B} = \{ \Psi_u(p), \Psi_v(p) \} \text{ is a basis for } T_pM.$$

We define

$$E = \Psi_u \cdot \Psi_u$$

$$F = \Psi_u \cdot \Psi_v = \Psi_v \cdot \Psi_u$$

$$G = \Psi_v \cdot \Psi_v$$

} components of the First Fundamental Form wrt $\mathcal{B}$.

Quickly $\mathrm{I}_p : T_pM \to \mathbb{R}$, $\quad \mathrm{I}_p(w) = \|w\|^2$

<u>First Fundamental Form</u>

If one writes $w = c_1 \vec{\Psi}_u + c_2 \vec{\Psi}_v$ at p.

then

$$\mathrm{I}_p(w) = \|w\|^2 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Classical Notation for I^{st}FF

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

Defn $\mathbb{II}^{nd}$ FF (2nd Fundamental Form)

$$\mathbb{II}_p : T_pM \longrightarrow \mathbb{R} \qquad \mathbb{II}_p(w) = S_p(w)\cdot w$$

$w = (c_1\Psi_u + c_2\Psi_v)(p) \in T_pM$

$\mathcal{B} = \{\Psi_u, \Psi_v\}$ basis of T_pM wrt Ψ param.

$$\mathbb{II}_p(w) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Components of the second Fundamental Form

$$\begin{cases} \ell = S_p(\Psi_u)\cdot\Psi_u = -\nabla_{\Psi_u}\mathcal{U}\cdot\Psi_u = \mathcal{U}\cdot\Psi_{uu} \\ m = S_p(\Psi_u)\cdot\Psi_v = -\nabla_{\Psi_u}\mathcal{U}\cdot\Psi_v = \mathcal{U}\cdot\Psi_{vu} \\ m = S_p(\Psi_v)\cdot\Psi_u = -\nabla_{\Psi_v}\mathcal{U}\cdot\Psi_u = \mathcal{U}\cdot\Psi_{uv} \\ n = S_p(\Psi_v)\cdot\Psi_v = -\nabla_{\Psi_v}\mathcal{U}\cdot\Psi_v = \mathcal{U}\cdot\Psi_{vv} \end{cases}$$

Weingarten Equations

Relation between I_p, II_p, S_p.

$$\varphi : D \to M, \quad \varphi(D) \ni p, \quad \mathcal{B} = \{\varphi_u, \varphi_v\} \text{ at } p.$$

$$\left.\begin{array}{l} S_p(\varphi_u) = a\varphi_u + b\varphi_v \\ S_p(\varphi_v) = c\varphi_u + d\varphi_v \end{array}\right\} \text{ for some } a,b,c,d \in \mathbb{R}.$$

$$[S_p]_{\mathcal{B}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Recall
$E = \varphi_u \cdot \varphi_u$
$F = \varphi_u \cdot \varphi_v$
$G = \varphi_v \cdot \varphi_v$

$$l = S_p(\varphi_u)\cdot\varphi_u = (a\varphi_u + b\varphi_v)\cdot\varphi_u = a\cdot E + bF$$

$$m = S_p(\varphi_u)\cdot\varphi_v = (a\varphi_u + b\varphi_v)\cdot\varphi_v = a\cdot F + bG$$

$$m = S_p(\varphi_v)\cdot\varphi_u = (c\varphi_u + d\varphi_v)\cdot\varphi_u = c\cdot E + d\cdot F$$

$$n = S_p(\varphi_v)\cdot\varphi_v = (c\varphi_u + d\varphi_v)\cdot\varphi_v = c\cdot F + dG.$$

$$(*) \quad \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{[S_p]^T}\cdot\underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{[I_p]} = \underbrace{\begin{bmatrix} l & m \\ m & n \end{bmatrix}}_{[II_p]}.$$

$$\det[I] = EG - F^2 \underset{\text{basic linear Algebra.}}{=} \|\varphi_u \times \varphi_v\|^2 \underset{\text{regular } \varphi.}{\neq} 0$$

$$[S_p]^T = [II_p][I_p]^{-1}$$

$$[S_p] = \left(([I_p]^{-1})^T\right) \cdot [II_p]^T$$

$$= \left(([I_p]^T)^{-1}\right) \cdot [II_p]$$

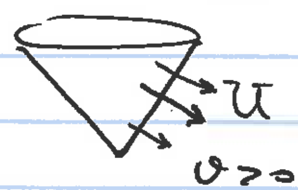
$$[S_p] = [I_p]^{-1}[II_p]$$

$[I]^T = [I]$

$[II]^T = [II]$

but $[S_p]$ may not be symmetric.

Ex. Exc. 2.3.9 + more.



$$\psi = (v\cos u,\ v\sin u,\ v)$$

$$\psi_u = (-v\sin u,\ v\cos u,\ 0)$$

$$\psi_v = (\cos u,\ \sin u,\ 1)$$

$$N = \psi_u \times \psi_v = (v\cos u,\ v\sin u,\ -v)$$

$$|N| = \sqrt{2}\, v$$

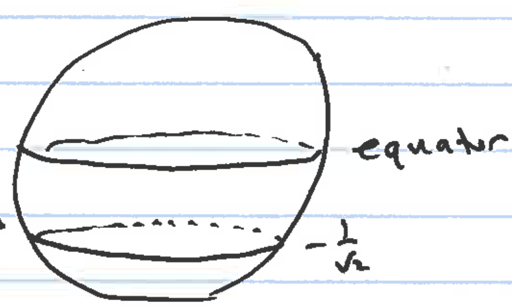
$$U = (\cos u,\ \sin u,\ -1)\frac{1}{\sqrt{2}}$$

Image of U

($-45°$ parallel.)

it is a curve!

(not always)



$$E = \Psi_u \cdot \Psi_u = v^2$$

$$F = \Psi_u \cdot \Psi_v = \Psi_v \cdot \Psi_u = 0$$

$$G = \Psi_v \cdot \Psi_v = 2$$

$$\Psi_{uu} = (-v\cos u, -v\sin u, 0)$$

$$\Psi_{vu} = \Psi_{uv} = (-\sin u, \cos u, 0)$$

$$\Psi_{vv} = (0, 0, 0)$$

$$U = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1).$$

$$l = U \cdot \Psi_{uu} = -\frac{v}{\sqrt{2}}$$

$$m = U \cdot \Psi_{uv} = 0$$

$$n = U \cdot \Psi_{vv} = 0$$

$$[S_p] = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} l & m \\ m & n \end{bmatrix}$$

$$= \begin{bmatrix} v^2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{v}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix}$$
if $ad \neq 0$

$$= \begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{v}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{v\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}.$$