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2.3 Linear Algebra review

Defn Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map.
L is called symmetric if
$\forall v, w \in \mathbb{R}^n \qquad v \cdot L(w) = L(v) \cdot w$

Obs ① If $\{u_1, u_2 \dots u_n\}$ is an orthonormal basis of $\mathbb{R}^n$
the the matrix of L wrt this basis is
a symmetric matrix.

Ex
$L(e_1) = e_1$
$L(e_2) = 2e_2$
$L(x,y) = (x, 2y)$

$$[L]_{\{e_1, e_2\}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

L is a symmetric linear map. $\Longleftarrow$ $\{e_1, e_2\}$ o.n. + symmetric matrix

However, if we use a different basis $\begin{cases} v_1 = (1,0) \\ v_2 = (1,1) \end{cases}$

$L(v_1) = (1,0) = v_1$
$L(v_2) = (1,2) = 2v_2 - v_1$

$$[L]_{\{v_1, v_2\}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}}_{\text{not symmetric matrix.}}$$

Defn A matrix A is called symmetric if

$$A^T = A$$

transpose

Principal Axis Thm:

For every symmetric matrix A ($A^T = A$), there exists an orthonormal basis $\{u_1 \dots u_n\}$ and an orthogonal matrix P ($PP^T = I = P^TP$) s.t.

$$P^TAP = D = \text{diagonal matrix}$$

and all eigenvalues of A are real, where

$$P = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n].$$

as columns

In other words, if $L: \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear map, then $\exists$ an orthonormal basis $\mathcal{B} = \{u_1 \dots u_n\}$ s.t.

$$[L]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}, \quad \lambda_i \in \mathbb{R}.$$

In short, every symmetric matrix is orthogonally diagonalizable.

Review/Prop/Def Given any $n \times n$ matrix A

$\det(\lambda I - A)$ characteristic polynomial.

$0 = \det(\lambda I - A)$ roots are called eigenvalues $\lambda_1, \lambda_2 \dots \lambda_n$.

$$\det(\lambda I - A) = \lambda^n - \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} \pm \cdots (-1)^i \sigma_i \lambda^{n-i} + \cdots + (-1)^n \sigma_n.$$

$\sigma_1 \ldots \sigma_n$ are called characteristic values.
$\lambda_1 \ldots \lambda_n$ are the eigenvalues

$$\sigma_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace } A.$$

$$\sigma_n = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A.$$

$$\sigma_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

(2) $\sigma_k(A) = \sigma_k(P^{-1}AP)$ $\forall k$, $\forall$ invertible matrix P.

A and $P^{-1}AP$ are called similar matrices.

Recall $L: \mathbb{R}^n \to \mathbb{R}^n$ linear
$\mathcal{B}_1, \mathcal{B}_2$ be 2 bases for $\mathbb{R}^n$

$$[L]_{\mathcal{B}_1} = P^{-1}[L]_{\mathcal{B}_2} P.$$

(3) The characteristic values associated to a linear map L (e.g. trace, det, etc. σ_k) are independent of the choice of the bases used in the calculation of the matrix of the linear map L.

Prop ① $S_p : T_pM \to T_pM$ is a symmetric linear map.

that is, $\forall$ parametrization ψ of M about p

$$S_p(\psi_u)\cdot\psi_v = S_p(\psi_v)\cdot\psi_u .$$

② $S_p(\psi_u)\cdot\psi_v = \mathcal{U}\cdot\psi_{vu}$

$S_p(\psi_u)\cdot\psi_u = \mathcal{U}\cdot\psi_{uu}$ ✓

$S_p(\psi_v)\cdot\psi_u = \mathcal{U}\cdot\psi_{uv}$

$S_p(\psi_v)\cdot\psi_v = \mathcal{U}\cdot\psi_{vv}$

Proof: Recall S_p symmetric means:

$\forall v,w \qquad S_p(v)\cdot w = S_p(w)\cdot v$

All we need to show this is true for a basis $\{\psi_u,\psi_v\}$ at p.

$S_p(\psi_u)\cdot\psi_u = S_p(\psi_u)\cdot\psi_u$ obvious.

$S_p(\psi_v)\cdot\psi_v = S_p(\psi_v)\cdot\psi_v$ "

But we Need: $S_p(\psi_u)\cdot\psi_v = S_p(\psi_v)\cdot\psi_u$

$$U = \frac{\Psi_u \times \Psi_v}{|\Psi_u \times \Psi_v|}$$

$$[U \cdot \Psi_u] \equiv 0$$

normal tangent

$$0 = \vec{\Psi}_u \cdot [U \cdot \Psi_u] = \vec{\Psi}_u \left(\sum_{i=1}^{3} U_i \cdot \frac{\partial x_i}{\partial u} \right)$$

$$U = (U_1, U_2, U_3), \quad \Psi = (x_1, x_2, x_3) \Big\} \quad U \cdot \Psi = U_1 x_1 + U_2 x_2 + U_3 x_3$$

$$\Psi_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u} \right) \quad \Psi_{uu} = \left(\frac{\partial^2 x_1}{\partial u^2}, \frac{\partial^2 x_2}{\partial u^2}, \frac{\partial^2 x_3}{\partial u^2} \right)$$

$$0 = \sum_{i=1}^{3} \vec{\Psi}_u [U_i] \cdot \frac{\partial x_i}{\partial u} + U_i \cdot \vec{\Psi}_u \left[\frac{\partial x_i}{\partial u} \right]$$

$$= \sum_{i=1}^{3} \left(\frac{\partial}{\partial u} (U_i(u,v)) \cdot \frac{\partial x_i}{\partial u} + U_i \cdot \frac{\partial^2 x_i}{\partial u^2} \right)$$

$$0 = \nabla_{\Psi_u} U \cdot \Psi_u + U \cdot \Psi_{uu}$$

$$0 = -S_p(\Psi_u) \cdot \Psi_u + U \cdot \Psi_{uu}$$

$$\Psi_u \cdot S_p(\Psi_u) = U \cdot \Psi_{uu}.$$

Similarly

$$0 = \psi_v\left[\overbrace{U\cdot\psi_u}^{0}\right] = \vec{\psi}_v\left(\sum_{i=1}^{3} U_i\cdot\frac{\partial x_i}{\partial u}\right)$$

$$= \sum_{i=1}^{3} \vec{\psi}_v[U_i]\cdot\frac{\partial x_i}{\partial u} + U_i\cdot\psi_v\left[\frac{\partial x_i}{\partial u}\right]$$

$$= \sum_{i=1}^{3} \frac{\partial}{\partial v} U_i(u,v)\cdot\frac{\partial x_i}{\partial u} + U_i\cdot\frac{\partial}{\partial v}\left[\frac{\partial x_i}{\partial u}\right]$$

$$= \left(\frac{\partial}{\partial v}U_1, \frac{\partial}{\partial v}U_2, \frac{\partial}{\partial v}U_3\right)\cdot\left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right) +$$

$$+ (U_1, U_2, U_3)\cdot\left(\frac{\partial^2 x_1}{\partial v\partial u}, \frac{\partial^2 x_2}{\partial v\partial u}, \frac{\partial^2 x_3}{\partial v\partial u}\right)$$

$$= \left(\psi_v[U_1], \psi_v[U_2], \psi_v[U_3]\right)\cdot\psi_u +$$

$$+ U\cdot\frac{\partial^2}{\partial v\partial u}\psi$$

$$= \nabla_{\psi_v}(U)\cdot\psi_u + U\cdot\psi_{uv}$$

$$0 = -S_p(\psi_v)\cdot\psi_u + U\cdot\psi_{uv}$$

$$\left.\begin{aligned} S_p(\psi_v)\cdot\psi_u &= U\cdot\psi_{uv} \\ S_p(\psi_u)\cdot\psi_v &= U\cdot\psi_{vu} \end{aligned}\right] \quad \text{but } \psi_{uv} = \psi_{vu} \text{ if } \psi\in C^2.$$

$$\Longrightarrow S_p(\psi_v)\cdot\psi_u = \psi_v\cdot S_p(\psi_u)$$

$\Longrightarrow$ S_p is a symmetric linear map.