3.3 To Conclude

Surfaces of revolution with $K \equiv +1$.

$$K = -\frac{h''}{h} \Leftarrow \left\{ \begin{array}{l} (g,h) \text{ profile} \\ g'^2 + h'^2 = 1 \equiv \sigma^2 \end{array} \right.$$

$$1 = -\frac{h''}{h}$$

$$h'' + h = 0$$

$$h(t) = A\cos t + B\sin t = c \cdot \cos(t + \phi_0)$$

$$c^2 = A^2 + B^2.$$

WLOG $\phi_0 = 0$, $h(t) = c \cdot \cos t$

$h'(t) = -c \sin t$

$$g' = \pm\sqrt{1 - (h'(t))^2} = \pm\sqrt{1 - c^2 \sin^2 t}$$

$$g = \int \underbrace{\sqrt{1 - c^2 \sin^2 t}}\, dt$$

We need $1 - c^2 \sin^2 t \geq 0$

$$1 \geq c^2 \sin^2 t$$

$$\frac{1}{c^2} \geq \sin^2 t$$

$$\frac{1}{c} \geq |\sin t|$$

$\Longrightarrow$ Domain needs to be restricted when $c < 1$.

e.g: $c = 2$ $\quad 1 - 4\sin^2 t \geq 0$

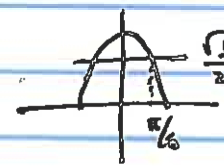
$$1 \geq 4 \sin^2 t$$

$$\frac{1}{4} \geq \sin^2 t$$

$$\frac{1}{2} \geq \sin t \geq -\frac{1}{2}$$

$$-\frac{\pi}{6} < t < \frac{\pi}{6}.$$

$$h(t) = 2\cos t$$



$$2 \cdot \frac{\sqrt{3}}{2} \leq h(t) \leq 1 \cdot 2$$

$$\sqrt{3} \leq h(t) \leq 2.$$

$$\varphi(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$$

See p153

$c > 1$

$K \equiv 1$

edges are singular, Surface cannot be extended a larger surface of $K \equiv +1$

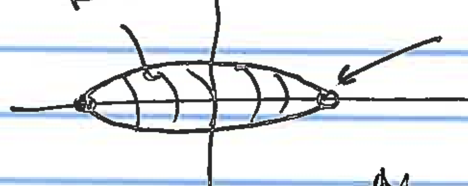
singular

incomplete

non-compact

$K \equiv 1$

$c < 1$



conical singularity

not singular

THM: Liebmann $\quad K \equiv 1$, M compact $\Rightarrow M = S^2_1$.

3.4

<u>Intrinsic</u> Any quantity of a given surface which can be calculated from the first fundamental form, (without using second fundamental form)

<u>Extrinsic</u> Any quantity essentially depends on the 2^{nd} F.F, as well as 1^{st} F.F.

<u>Intrinsic</u>: Length of curves
Area of regions
Angles
Shortest curves between pts.

<u>Extrinsic</u>: H mean curvature, principal curvature

<u>Ex</u> cylinder $\psi(u,v) = (\cos u, \sin u, v)$ C
plane $\phi(u,v) = (u, v, 0)$. P

$$\left.\begin{aligned} E_\psi &= 1 = E_\phi \\ F_\psi &= 0 = F_\phi \\ G_\psi &= 1 = G_\phi \end{aligned}\right\} \text{identical } I^{st} \text{ F.}$$

Not identical H's.

$$\left.\begin{aligned} H_C &\neq 0 \\ H_P &= 0 \end{aligned}\right\} H \text{ is not intrinsic}$$

What about K?

$$K = \frac{ln - m^2}{EG - F^2}$$ is NOT the only formula for K.

1826 Gauss Theorema Egregium

"Remarkable, Singular, Eminent (outrageous?)

Theorema Egregium K is intrinsic.

STEP 1 Christoffel Symbols

Defn Let $\varphi(u,v)$ be a local parametrization of a surface M about $p \in M$.

At $p \in M$, $\{\varphi_u(p), \varphi_v(p), U(p)\}$ forms a basis for $T_p\mathbb{R}^3$

Basis $\Rightarrow$ $\exists\, \Gamma^i_{jk}$ such that

$$\left\{\begin{aligned} \varphi_{uu} &= \Gamma^u_{uu}\varphi_u + \Gamma^v_{uu}\varphi_v + lU \\ \varphi_{uv} &= \Gamma^u_{uv}\varphi_u + \Gamma^v_{uv}\varphi_v + mU \\ \varphi_{vv} &= \Gamma^u_{vv}\varphi_u + \Gamma^v_{vv}\varphi_v + nU \end{aligned}\right\} \quad (*)$$

$\{\Gamma^i_{jk}\}$ are called Christoffel symbols of the first kind.

Prop 1 Christoffel Symbols are intrinsic

Proof: How to find Γ^i_{jk}: By ⊛ on page 4 obtain RHS:

$$\tfrac{1}{2}E_u = \psi_{uu}\cdot\psi_u = \Gamma^u_{uu}\cdot E + \Gamma^v_{uu}\cdot F + 0 \quad (\#1)$$

$$F_u - \tfrac{1}{2}E_v = \psi_{uu}\cdot\psi_v = \Gamma^u_{uu}\cdot F + \Gamma^v_{uu} G + 0 \quad (\#2)$$

$$\tfrac{1}{2}E_v = \psi_{uv}\cdot\psi_u = \Gamma^u_{uv}\cdot E + \Gamma^v_{uv} F + 0 \quad (\#3)$$

$$\tfrac{1}{2}G_u = \psi_{uv}\cdot\psi_v = \Gamma^u_{uv} F + \Gamma^v_{uv} G + 0 \quad (\#4)$$

$$F_v - \tfrac{1}{2}G_u = \psi_{vv}\cdot\psi_u = \Gamma^u_{vv} E + \Gamma^v_{vv} F + 0 \quad (\#5)$$

$$\tfrac{1}{2}G_v = \psi_{vv}\cdot\psi_v = \Gamma^u_{vv} F + \Gamma^v_{vv} G + 0 \quad (\#6)$$

LHS:

$$E_u = \frac{d}{du}(\underbrace{\psi_u\cdot\psi_u}_{E}) = 2\psi_u\cdot\psi_{uu} \quad \text{for eq}^n\ \#1$$

$$F_u = \frac{d}{du}(\underbrace{\psi_u\cdot\psi_v}_{F}) = \psi_{uu}\cdot\psi_v + \psi_u\cdot\psi_{vu}$$

$$= \psi_{uu}\cdot\psi_v + \psi_u\cdot\psi_{uv}$$

$$E_v = \frac{\partial}{\partial v}(\underbrace{\psi_u\cdot\psi_u}_{E}) = 2\psi_u\cdot\psi_{uv}$$

$\psi_{uu}\cdot\psi_v = F_u - \tfrac{1}{2}E_v$. #3–6 are similar.

#1 – #6 can be put into matrix equation

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{2} E_u & \frac{1}{2} E_v & F_v - \frac{1}{2} G_u \\ F_u - \frac{1}{2} E_v & \frac{1}{2} G_u & \frac{1}{2} G_v . \end{bmatrix}}$$

$$\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix} = \underbrace{\overbrace{\frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}}^{[I_p]^{-1}}\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}$$

$$\underbrace{\text{everything is calculable from } E, F, G}_{\text{Intrinsic.}}$$

$\Longrightarrow$ Γ^k_{ij} are intrinsic.

<u>Obs:</u> $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\forall i, j, k$.

Main idea

$$E, F, G \xrightarrow[\text{did it Prop. I}]{\partial} \Gamma^k_{ij} \xrightarrow[\text{Next step.}]{\partial} K$$