Includes Review from 11/3/17 ①

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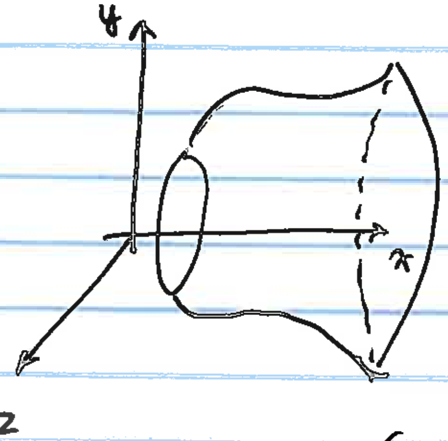
③.3 Surfaces of revolution

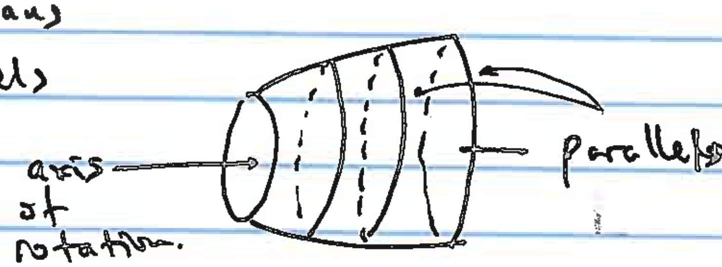


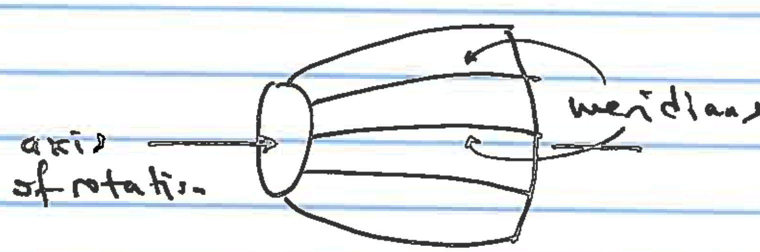
profile curve $(g(u), h(u))$ $h(u) \neq 0$
regular

rotate about x-axis

$$\varphi(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

$\varphi(u, v_0)$ meridians
$\varphi(u_0, v)$ parallels





$$\varphi_u = (g',\ h'\cos v,\ h'\sin v)$$

$$\varphi_v = (0,\ -h\sin v,\ h\cos v)$$

$$\varphi_u \times \varphi_v = (hh',\ -g'h\cos v,\ -g'h\sin v)$$

$$= h(h',\ -g'\cos v,\ -g'\sin v)$$

$$\|\varphi_u \times \varphi_v\| = h\underbrace{\sqrt{(h')^2 + (g')^2}}_{\sigma} = h\sigma \neq 0$$

σ is speed of profile curve

$$U = \frac{1}{\sigma}\left(h', -g'\cos\theta, -g'\sin\theta \right)$$

Caution if $g' > 0$ then inward normal

$$\Psi_{uu} = \left(g'', h''\cos\theta, h''\sin\theta \right)$$

$$\Psi_{u\theta} = \left(0, -h'\sin\theta, +h'\cos\theta \right)$$

$$\Psi_{\theta\theta} = \left(0, -h\cos\theta, -h\sin\theta \right)$$

$$\ell = \Psi_{uu} \cdot U = \frac{1}{\sigma}\left(h'g'' - h''g' \right)$$

$$m = 0$$

$$n = \frac{hg'}{\sigma}.$$

$$\mathrm{II} = \begin{bmatrix} \ell & 0 \\ 0 & n \end{bmatrix}.$$

$$E = \sigma^2$$

$$F = 0$$

$$G = h^2$$

$$\mathrm{I} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}.$$

$$[S_p]^T = [\mathrm{II}_p][\mathrm{I}_p]^{-1} = \begin{bmatrix} \ell & 0 \\ 0 & n \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\ell}{E} & 0 \\ 0 & \frac{n}{G} \end{bmatrix}$$

Since this matrix is diagonal, $[S_p]^T = [S_p]$.

$$K = \frac{\ell}{E}\cdot\frac{n}{G} = \underbrace{\frac{1}{\sigma^3}\left(h'g''-h''g'\right)}_{k_\mu}\cdot\underbrace{\frac{hg'}{\sigma}}_{k_\pi}\cdot\frac{1}{h^2}$$

Gaussian $$\boxed{K = \frac{g'\left(h'g''-h''g'\right)}{h\cdot\sigma^4}}$$

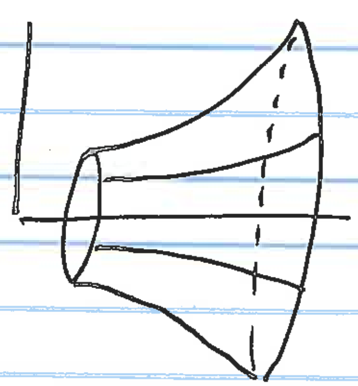
Principal Curvatures

$$k_\pi = \frac{n}{G} = \frac{g'}{h\sigma}$$

circular

$$k_\mu = \frac{\ell}{E} = \frac{1}{\sigma^3}\left(h'g''-h''g'\right) \longleftarrow$$ curve curvature of the profile curve.

meridian direction



(1) All meridians are normal sections by a plane thru axis of rotation.

(2) All meridians are rotations of the profile curve, and hence all are congruent.

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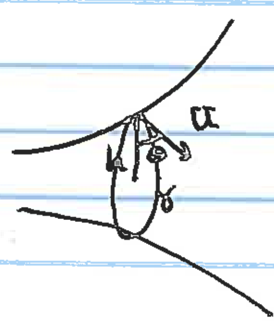
NOT True for Parallels:



Curve curvature of a parallel is $1/h$.

$$k_\pi = k_\gamma\cdot\cos\theta \overset{?}{=} \frac{1}{h}\cdot\frac{g'}{\sigma} = \frac{n}{G}.$$

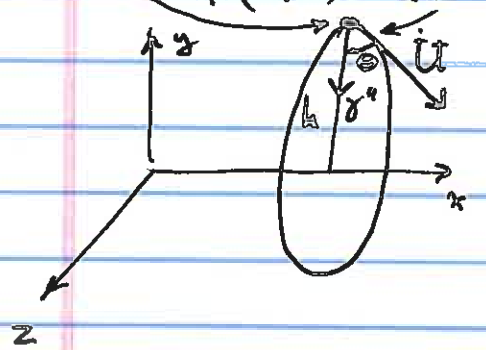
$\theta = \measuredangle(\mathcal{U}(p), \gamma''(p))$ where γ is a curve thru p, in our case it is a circle of radius h.

$\gamma(v) = (g(u), h(u)\cos v, h(u)\sin v)$

$\underline{u \text{ fixed.}}$

$v=0$

$\varphi(u,0) = (g(u), h(u), 0)$



$\dfrac{\gamma''(p)}{|\gamma''(p)|}$ points towards the center of the circle.

$$\mathcal{U}(u,0) = \frac{1}{\sigma}(h', -g', 0)$$

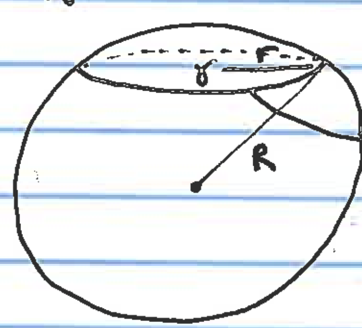
$$\gamma''/|\gamma''| = (0,-1,0)$$

$$\mathcal{U}(u,0)\cdot\frac{\gamma''}{|\gamma''|}(p) = \frac{g'}{\sigma} = \cos\theta$$

$$k_\pi = k_\gamma \cdot \cos\theta = \frac{1}{h}\cdot\frac{g'}{\sigma} = \frac{n}{G}.$$

Obs $\pm k_\pi = k_\gamma$ when $h' = 0$ ($\Rightarrow \sigma = |g'|$)

Ex.



S^2_R

$k_\gamma = \frac{1}{r} \neq \underbrace{\text{normal curvature}}_{\pm\frac{1}{R}}$

unless parallel = equator

i.e. $r = R$.

$$K = \frac{g'(h'g'' - h''g')}{h\sigma^4}$$

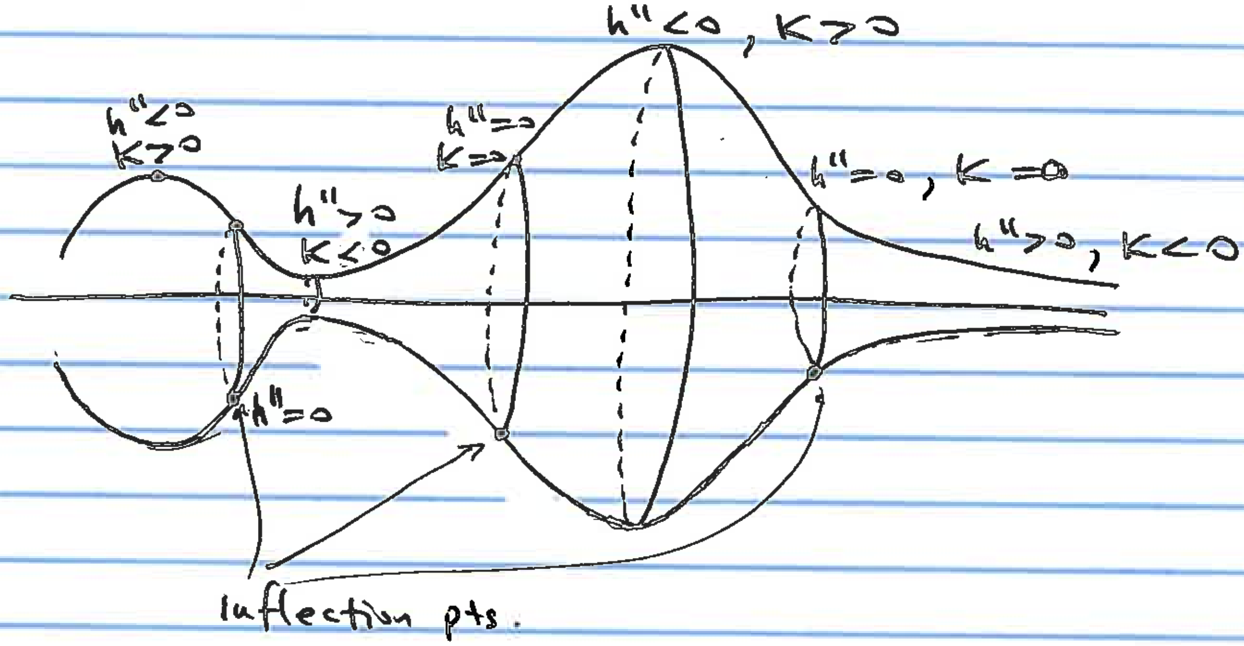
Profile (g, h)
$\sigma^2 = g'^2 + h'^2$

Special Cases ① Profile curve is an explicit graph.

$(g, h) = (u, h(u))$ i.e. $g(u) = u$.
$g' = 1$
$g'' = 0$

$$K = -\frac{h''}{h} \cdot \frac{1}{\sigma^4}$$

$h, \sigma > 0 \implies K$ and h'' have opposite signs.



Exc (3.3.8) Special case 2 $\sigma = 1$

profile curve is parametrized w.r.t. arclength.

$$1 = \sigma^2 = g'^2 + h'^2$$

$$0 = 2g'g'' + 2h'h'' \longrightarrow g'g'' = -h'h''$$

$$K = \frac{g'(h'g'' - h''g')}{h \cdot \sigma^4} = \frac{g'h'g'' - g'h''g'}{h}$$

$$= \frac{-h'h'h'' - g'g'h''}{h} = -\frac{h''}{h}.$$

$$\boxed{K = -\frac{h''}{h}} \Longleftrightarrow h'' + Kh = 0$$

Exc 3.3.12 Surfaces of revolution with $K \equiv -1$.

We need $(g(u), h(u))$, $\sigma^2 = 1 = h'^2 + g'^2$

$$-1 = -\frac{h''}{h}.$$

Need Not both A & $B = 0$.

$$h'' - h = 0.$$

$$h = Ae^{u} + Be^{-u}$$

$$h' = Ae^{u} - Be^{-u}.$$

need $|h'| \leq 1$

$$g' = \pm\sqrt{1 - (h')^2} = \pm\sqrt{1 - (Ae^{u} - Be^{-u})^2}$$

A, B not both 0 $\Longrightarrow$ not defined for all u.

Try the easiest case:

Let $A = 1$, $B = 0$ $\Big\{$ $h = e^u$

$h' = e^u$

$$1 = g'^2 + h'^2 = g'^2 + e^{2u}$$

$$g'^2 = 1 - e^{2u}$$

$$g' = \pm\sqrt{1 - e^{2u}} \qquad \text{need } u \le 0$$

$$g = \int \sqrt{1 - e^{2u}}\, du = ?$$

Substitute $e^u = \operatorname{sech} w$ $\quad \sqrt{1 - e^{2u}} = \tanh w$

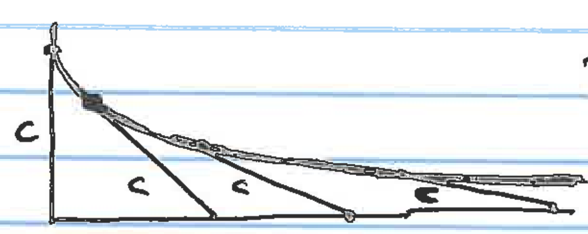
$$u = \ln \operatorname{sech} w = -\ln \cosh w$$

$$du = -\frac{\sinh w}{\cosh w} = -\tanh w$$

$$g = \int \sqrt{1 - e^{2u}}\, du = \int -\tanh^2 w\, dw = \int 1 + \operatorname{sech}^2 w\, dw$$

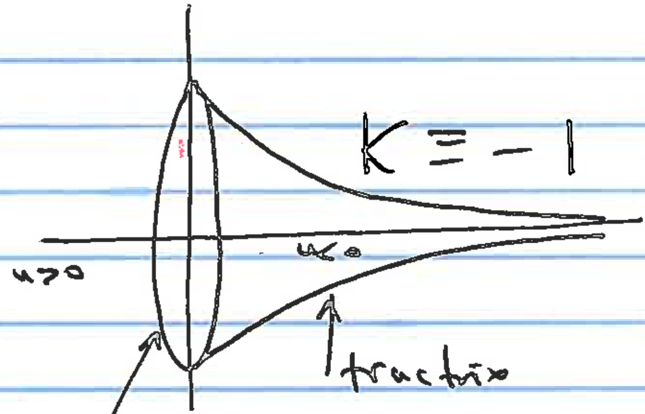
$$= w - \tanh w$$

$(g, h) = (w - \tanh w, \operatorname{sech} w)$ Called tractrix.



Dragging a big log of length c

$\varphi(u, \theta) = (u - \tanh u, (\operatorname{sech} u)(\cos\theta), (\operatorname{sech} u)(\sin\theta))$



Pseudosphere

Part of the hyperbolic space

non-removable singularity along circle,
It cannot be extended beyond $u=0$, with $K \equiv -1$

If one chooses different A, B, all surfaces obtained have singular boundaries, that is these surfaces are not extendible beyond them, with $K \equiv -1$.

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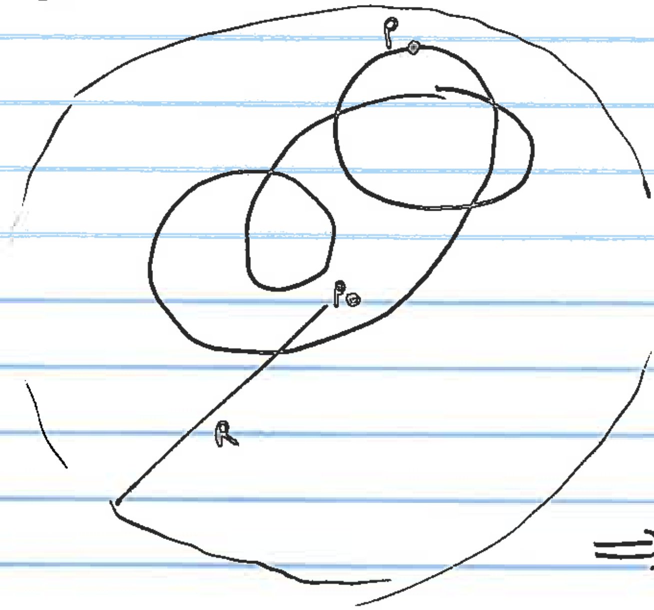
See p 154

<u>Theorem</u> Hilbert (1901) There doesn't exist a smooth complete (immersed) 2-surface of $K \equiv -1$ in $\mathbb{R}^3$.

Long Proof.

<u>But</u> Compact Case is easy to prove:

Recall Let γ be a closed C^2 curve in $\mathbb{R}^n$



Let

$\gamma \subseteq B_R(p_0)$

$B_R(p_0) = \{ x \in \mathbb{R}^n \mid |x - p_0| < R \}$

$\Longrightarrow \exists p \in \gamma, \quad \underset{\text{curve curvature}}{k_\gamma(p)} > \frac{1}{R}.$

Prop Let M be a closed, compact, regular 2-surface in $\mathbb{R}^3$.

Let $M \subseteq B_R(p_0) \subseteq \mathbb{R}^3$.

then $\exists \; p \in M, \quad K(p) \geq \frac{1}{R^2} > 0$

Conclusion Every compact surface in $\mathbb{R}^3$ has a pt of positive Gaussian curvature.

(compact $\Longrightarrow$ bounded

$\Longrightarrow M \subseteq B_R(p_0)$ for some p_0, R)