Nov 3, 2017

Curvatures of Ruled Surfaces

$$\Psi(u,v) = \beta(u) + v\delta(u) \qquad |\beta'| = 1$$
$$|\delta| = 1$$

Recall Ex Ruled Surfaces

- One sheeted Hyperboloid
- $z = xy$ saddle
- Helicoid

$K < 0$

- Cylinders
- Cones

$K = 0$

Exc. (3.2.18a) p117

$$\Psi_u = \beta' + v\delta'$$
$$\Psi_v = \delta$$

$$\Psi_u \times \Psi_v = (\beta' + v\delta') \times \delta = \beta' \times \delta + v(\delta' \times \delta)$$

$$W = \| \Psi_u \times \Psi_v \| = \| \beta' \times \delta + v(\delta' \times \delta) \|$$

$$U = \frac{1}{W}(\beta' \times \delta + v(\delta' \times \delta))$$

$$\Psi_{uu} = \beta'' + v\delta''$$
$$\Psi_{uv} = \delta'$$
$$\Psi_{vv} = 0$$

$$l = U \cdot \Psi_{uu} = \text{no need}$$

$$m = U \cdot \Psi_{uv} = \frac{1}{W}(\beta' \times \delta + v \underbrace{(\delta' \times \delta)) \cdot \delta'}_{\perp}$$

$$= \frac{1}{W}(\beta' \times \delta) \cdot \delta'$$

$$n = U \cdot \Psi_{vv} = 0$$

$$K = \frac{ln - m^2}{EG - F^2} = -\frac{1}{W^2}((\beta' \times \delta) \cdot \delta')^2 \cdot \frac{1}{W^2}$$

recall $EG - F^2 = \| \varphi_u \times \varphi_v \|^2$.

$$K = -\frac{((\beta' \times \delta) \cdot \delta')^2}{W^4} \leq 0.$$

Question: When is $K \equiv 0$ along a ruling

Obs 1. $((\beta' \times \delta) \cdot \delta')^2$ has no v but

$$W^2 = \| \beta' \times \delta + v(\delta' \times \delta) \|^2 \text{ does}$$

$\Rightarrow$ Along a ruling (line) either $K \equiv 0$ on all of the line

or $K < 0$ on all of the line

$\Big\}$ no mix along one line.

Prop:

$\underbrace{K \equiv 0 \text{ along } \varphi(u_0, v)}_{\text{ruling}} \Rightarrow$ (i) $\delta'(u_0) = 0$

OR

(ii) The ruling $\varphi(u_0, v)$ meets the line of striction at a singular pt when $\delta'(u_0) \neq 0$.

(i) is obvious, So suppose $\delta'(u_0) \neq 0$.

By HW Exc 2.1.24 we can reparametrize the surface s.t. $\beta' \cdot \delta' = 0$ (line of striction)

Recall $\delta' \perp \delta \Leftarrow |\delta| = 1$.

$\delta' \perp \beta' \Leftarrow (\beta' \cdot \delta' = 0)$

Either $\delta \times \beta' = 0$ or $\delta' \parallel \delta \times \beta'$ (both $\neq 0$)

Assumed $K = -\frac{((\beta' \times \delta) \cdot \delta')^2}{W^4} \equiv 0$ Can't happen

$$\delta \times \rho' = 0 \quad \text{at } u_0.$$

$$\delta \parallel \rho' \quad \text{at } u_0.$$

$$EG - F^2 = W^2 = \| \underbrace{\rho' \times \delta}_{0} + v(\delta' \times \delta) \|^2 = |v| \, \| \underbrace{\delta' \times \delta}_{\neq 0} \|$$

since $\delta \neq 0$
$\delta' \neq 0$
$\delta \perp \delta'$

When $v = 0$, i.e. line of striction
$\Psi_u \times \Psi_v = 0$. singular surface along the line of striction at $u = u_0$.

Even when you don't have a ruled surface:

<u>Prop</u> Let M be a regular surface, let L be a line $L \subseteq M$.

$$\forall p \in L, \quad K(p) \leq 0.$$

<u>Proof</u> Let $\vec{w}_0$ be a direction of L.

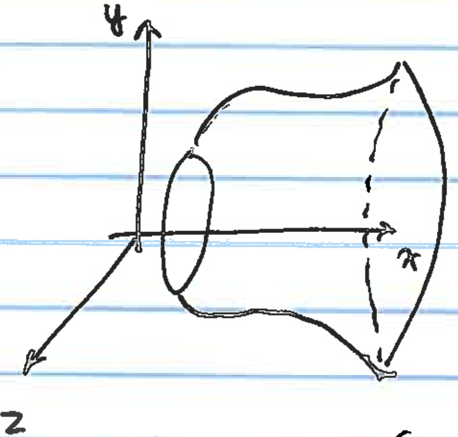
$$\mathrm{II}_p(\vec{w}_0) = 0$$

$$k_2 \leq \underbrace{\mathrm{II}_p(\vec{w}_0)}_{= 0} \leq k_1 \qquad \text{principal curvatures } k_1, k_2.$$

$$k_2 \leq 0 \leq k_1 \implies k_1 k_2 \leq 0.$$

11/6/17 lecture starts on page 3

3.3 Surfaces of revolution

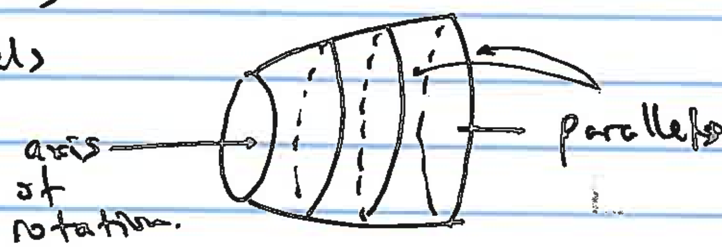


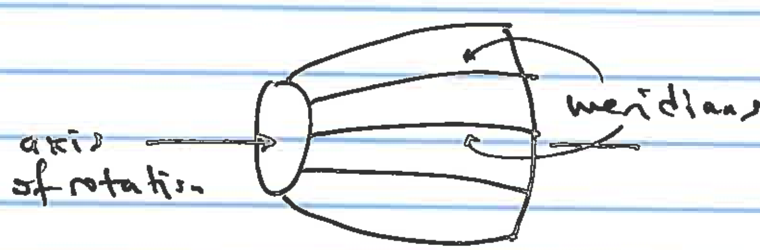
profile curve $(g(u), h(u))$ $h(u) \neq 0$
regular

rotate about x-axis

$$\psi(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

$\psi(u, v_0)$ meridians
$\psi(u_0, v)$ parallels





$$\psi_u = (g',\ h'\cos v,\ h'\sin v)$$

$$\psi_v = (0,\ -h\sin v,\ h\cos v)$$

$$\psi_u \times \psi_v = (hh',\ -g'h\cos v,\ -g'h\sin v)$$

$$= h(h',\ -g'\cos v,\ -g'\sin v)$$

$$\|\psi_u \times \psi_v\| = h\underbrace{\sqrt{(h')^2 + (g')^2}}_{\sigma} = h\sigma \neq 0$$

$\sigma \neq 0$ is speed of profile curve

$$U = \frac{1}{\sigma}\left(h', -g'\cos\theta, -g'\sin\theta \right)$$

Caution if $g' > 0$
then inward
normal

$$\Psi_{uu} = \left(g'', h''\cos\theta, h''\sin\theta \right)$$

$$\Psi_{u\theta} = \left(0, -h'\sin\theta, +h'\cos\theta \right)$$

$$\Psi_{\theta\theta} = \left(0, -h\cos\theta, -h\sin\theta \right)$$

$$l = \Psi_{uu} \cdot U = \frac{1}{\sigma}\left(h'g'' - h''g' \right)$$

$$m = 0$$

$$n = \frac{hg'}{\sigma}.$$

$$\mathrm{II} = \begin{bmatrix} l & 0 \\ 0 & n \end{bmatrix}.$$

$$E = \sigma^2$$

$$F = 0$$

$$G = h^2$$

$$\mathrm{I} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}.$$

$$[S_p]^T = [\mathrm{II}_p][\mathrm{I}_p]^{-1} = \begin{bmatrix} l & 0 \\ 0 & n \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}^{-1} = \begin{bmatrix} \frac{l}{E} & 0 \\ 0 & \frac{n}{G} \end{bmatrix}$$

Since this matrix is diagonal, $[S_p]^T = [S_p]$.

$$K = \frac{\ell}{E}\cdot\frac{n}{G} = \underbrace{\frac{1}{\sigma^3}\left(h'g''-h''g'\right)}_{k_\mu}\cdot\underbrace{\frac{hg'}{\sigma}\cdot\frac{1}{h^2}}_{k_\pi}$$

Gaussian $\boxed{K = \dfrac{g'(h'g''-h''g')}{h\cdot\sigma^4}}$

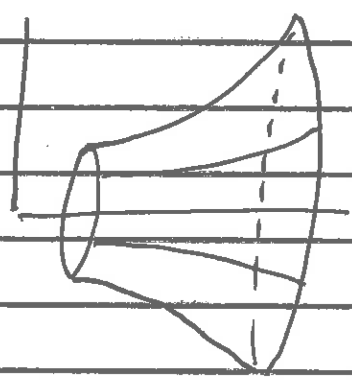
Principal Curvatures

$$k_\pi = \frac{n}{G} = \frac{g'}{h\sigma}$$

circular

$$k_\mu = \frac{\ell}{E} = \frac{1}{\sigma^3}\left(h'g''-h''g'\right) \longleftarrow$$ curve curvature of the profile curve.

meridian direction



(1) All meridians are normal sections by a plane thru axis of rotation.

(2) All meridians are rotations of the profile curve, and hence all are congruent.

11/6/17 $\longrightarrow$