Nov 29, 2017

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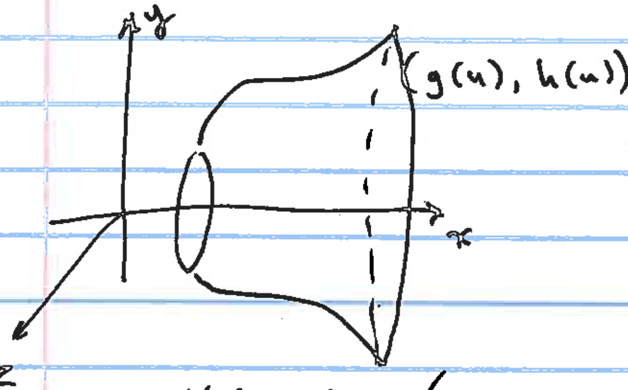
5.2 Geodesics of Surfaces of Revolution



$$\sigma = \sqrt{(g')^2 + (h')^2}$$

$$\varphi(u,v) = \left(g(u),\ h(u)\cos v,\ h(u)\sin v \right)$$

$$\left.\begin{aligned} E &= \sigma^2(u) \\ F &= 0 \\ G &= h^2(u) \end{aligned}\right\} \text{ Done earlier}$$

(Prop 1)
4.3

$$\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^u_{vv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_u \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{h^2} \end{bmatrix} \begin{bmatrix} \sigma\sigma' & 0 & -hh' \\ 0 & hh' & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma'}{\sigma} & 0 & -\frac{hh'}{\sigma^2} \\ 0 & \frac{h'}{h} & 0 \end{bmatrix}$$

Geodesic Eqns $\left.\begin{aligned} A(u,v) &= 0 \\ B(u,v) &= 0 \end{aligned}\right\}$ become

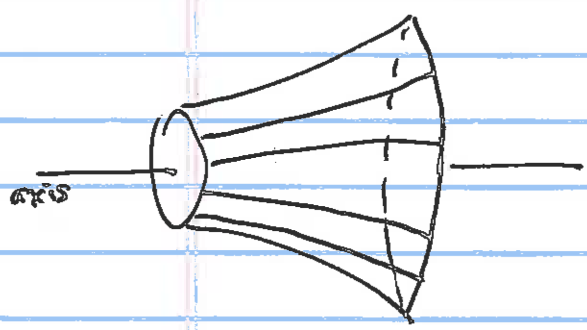
For the geodesic $\alpha(t) = \psi(u(t), v(t))$:

$$① \quad u'' + \frac{\sigma'}{\sigma}(u')^2 - \frac{hh'}{\sigma^2}(v')^2 = 0$$

$$② \quad v'' + 2\frac{h'}{h}u'v' = 0$$

} Geodesic Eq^ns for surfaces of revolution

<u>Prop 1</u> : All meridians of constant speed are geodesics (in a surface of Revolution)



$\psi(u, v_0)$ are the meridians.

constant

$$\Rightarrow \quad v' = 0$$
$$v'' = 0$$

Above ② is satisfied.

What remains to show is $u'' + \frac{\sigma'}{\sigma}(u')^2 = 0$ for ①

Assume Constant speed $\alpha(t) = \psi(u(t), v_0)$

$|\alpha'(t)| = c = 1$ WLOG PTO

why??

$\alpha(s)$ geodesic $\Rightarrow \alpha''_{tan} \equiv 0$

$\beta(s) = \alpha(cs) \qquad c \in \mathbb{R}$.

$\beta'(s) = c\,\alpha'(cs)$

$\beta''(s) = c^2 \alpha''(cs)$

$\alpha''_{tan} \equiv 0 \Rightarrow \beta''_{tan} \equiv 0$

} $\Rightarrow \alpha(s)$ geodesic $\Rightarrow \alpha(cs)$ is a geodesic $\forall c \in \mathbb{R}$

$\boxed{\text{sub lemma}}$

Continue proof of Prop 1.

$$\alpha(t) = \varphi(u(t), v_0)$$

$$\alpha'(t) = \varphi_u \cdot u' + \varphi_v \cdot \cancel{v'}^{\,0}$$

$$1 = |\alpha'(t)|^2 = |\varphi_u|^2 |u'|^2$$

$$1 = E \cdot (u')^2 = \underbrace{\sigma^2(u(t))}_{E} \cdot (u')^2$$

$$(u')^2 = \frac{1}{\sigma^2(u(t))}$$

$\frac{d}{dt}$

$$2 u' \cdot u'' = \frac{-2\sigma\sigma' \cdot u'}{\sigma^4}$$

$$0 = u'u'' + \frac{\sigma\sigma' u'}{\sigma^4} = u'\left(u'' + \frac{\sigma'}{\sigma} \cdot \frac{1}{\sigma^2}\right)$$

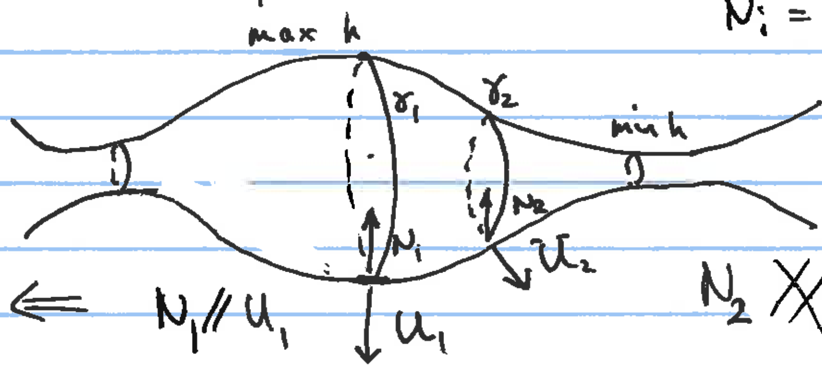
$$0 = u'\left(u'' + \frac{\sigma'}{\sigma} \cdot (u')^2\right)$$

$u' \neq 0$ $\qquad 0 = u'' + \frac{\sigma'}{\sigma}(u')^2$ ① is satisfied.

Prop I ✓

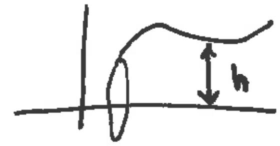
What about parallels?

N_i = principal normal of γ_i



$(\gamma_2'')_{tan} \neq 0$

γ_1 is a geodesic $\Longleftarrow$ $N_1 \parallel U_1$ $\qquad N_2 \not\parallel U_2 \Longrightarrow$ γ_2 is not a geodesic



Prop 2 $\quad \psi(u_0, v(t))$ is a geodesic

$$\iff \begin{cases} h'(u_0) = 0 & \text{and} \\ v(t) = at + b \end{cases}$$

Proof: $\quad u' = 0$

① $\quad -\dfrac{hh'}{\sigma^2}(v')^2 = 0.$

② $\quad v'' = 0$

(geodesic eq^{ns} when $u' \equiv 0 \equiv u''$, $u \equiv u_0$)

① $\quad 0 = v'' \iff v(t) = at + b.$

② $\quad -\dfrac{hh'}{\sigma^2}(v')^2 = 0 \implies v' = 0 \quad (\iff a = 0,$ pt geodesic$)$

or

$h = 0$ (not the case)

or.

$h' = 0$

singular surf. of revolution

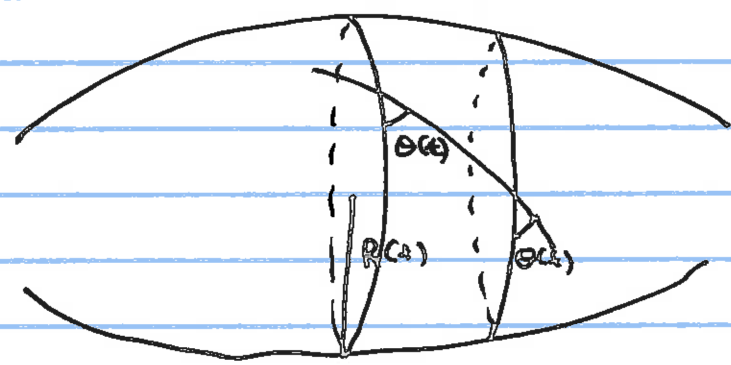
Also: $\quad h'(u_0) = 0 \implies -\dfrac{hh'}{\sigma^2}(v')^2 = 0$

$\psi(u_0, v(t))$ is a geodesic $\iff h'(u_0) = 0.$

(under assumption of $v(t) = at + b$)

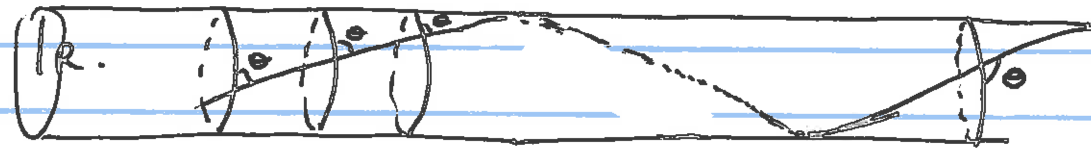
Thm: (Clairaut Relation)

Let $\alpha(t)$ be a geodesic of a surface of revolution, $|\alpha'(t)| = 1$.
Let $\theta = \measuredangle(\alpha'(t), \Psi_v(\alpha(t)))$, i.e. the angle $\alpha'(t)$ makes with the parallel thru $\alpha(t)$.
Then
$$R(\alpha(t))\cos\theta(t) \equiv \text{constant } \forall t.$$
where $R = h$ (distance to the rotation axis.)



Proof on Friday

Ex 1 Cylinder $\qquad R(t) = R_0$ constant



$$R(t)\cdot\cos\theta(t) = C_0$$

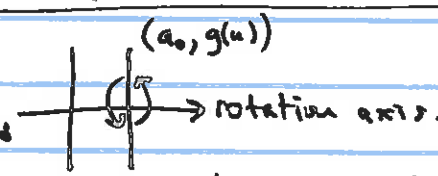
$$\Rightarrow \theta(t) \text{ is constant}$$

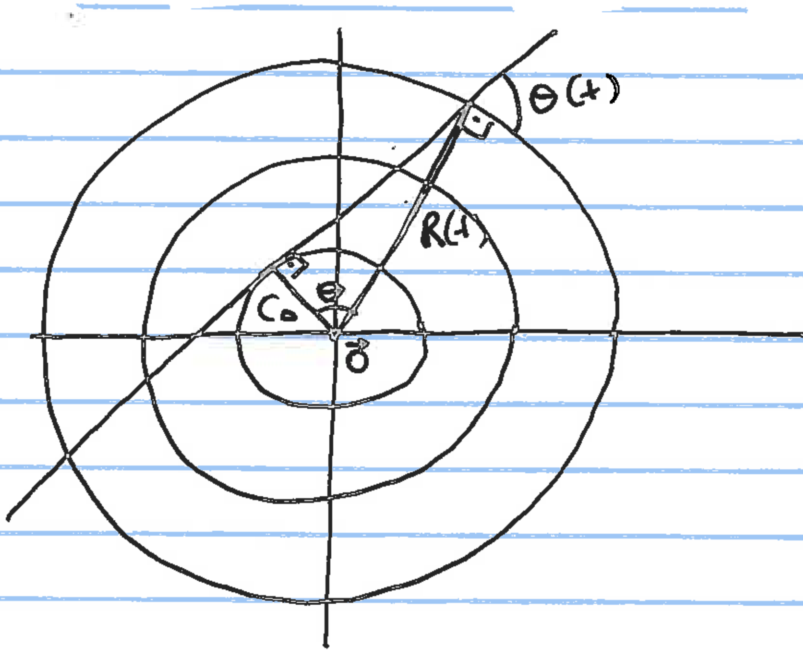
$\theta(t) \equiv 0$ circles (All parallels are geodesic since $R'(t) \equiv 0$

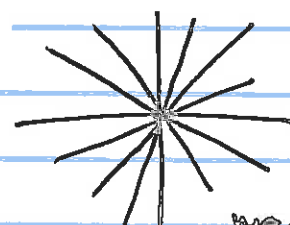
$\theta(t) \equiv \frac{\pi}{2}$ lines (meridians)

$0 < \theta < \frac{\pi}{2}$ helices

Ex 2 $\mathbb{R}^2$, which can be obtained by rotating a line perpendicular the rotation axis

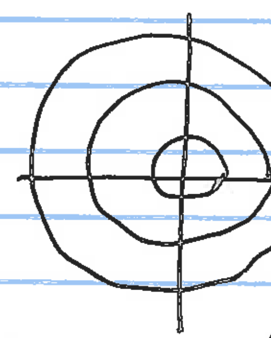






meridians are the lines thru $\vec{O}$

All are geodesics



In $\mathbb{R}^2$, none is a geodesic

parallels are the concentric circles centered at O

$$\cos\theta(t) = \frac{C_0}{R(t)}$$

$$R(t)\cdot\cos\theta(t) \equiv C_0 \ \forall t.$$