Nov 27, 20

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5.1 Continue

Thm I If γ is a shortest curve between p & q, then γ is a geodesic between p & q.

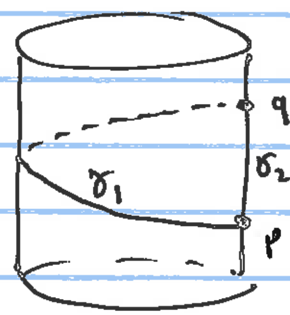
Shortest $\Longrightarrow$ Geodesic

Shortest $\cancel{\Longleftarrow}$ Geodesic Globally (A)

Shortest $\Longleftarrow$ Geodesic Locally. (B)

Ex 1

Cylinder



γ_1 helix

γ_2 vertical line

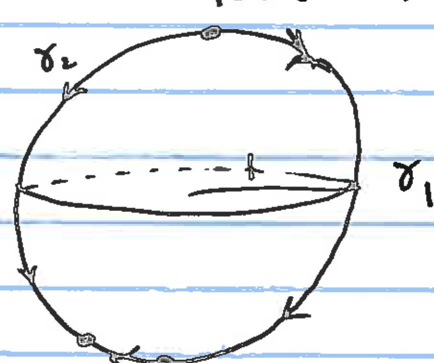
both geodesics.

γ_1 is not the shortest curve between p & q.

γ_2 is.

Ex 2

S^2_1



γ_1 from N to S, to P;

γ_2 from N to P.

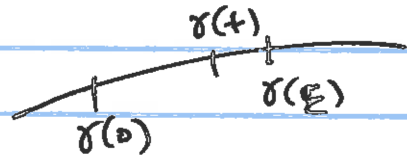
Both γ_1, γ_2 geodesics

γ_1 is not a shortest curve from N to P.

Thm. (MATH 6500)

(B) If $\gamma : [0,L] \to M$ is a geodesic $L>0$.

then $\exists$ $\varepsilon>0$, $\varepsilon \le L$ s.t. $\forall$ $t \in (0,\varepsilon)$ γ is the unique shortest curve from $\gamma(0)$ to $\gamma(t)$.



Recall If $T^2 \subseteq \mathbb{R}^3$ $\exists$ p_1, p_2, p_3 s.t.

$K(p_1) > 0$ proved i

$K(p_2) = 0$

$K(p_3) < 0$ $\Leftarrow$ (Ga

Gauss-Bonnet

Example Flat torus in $\mathbb{R}^4$

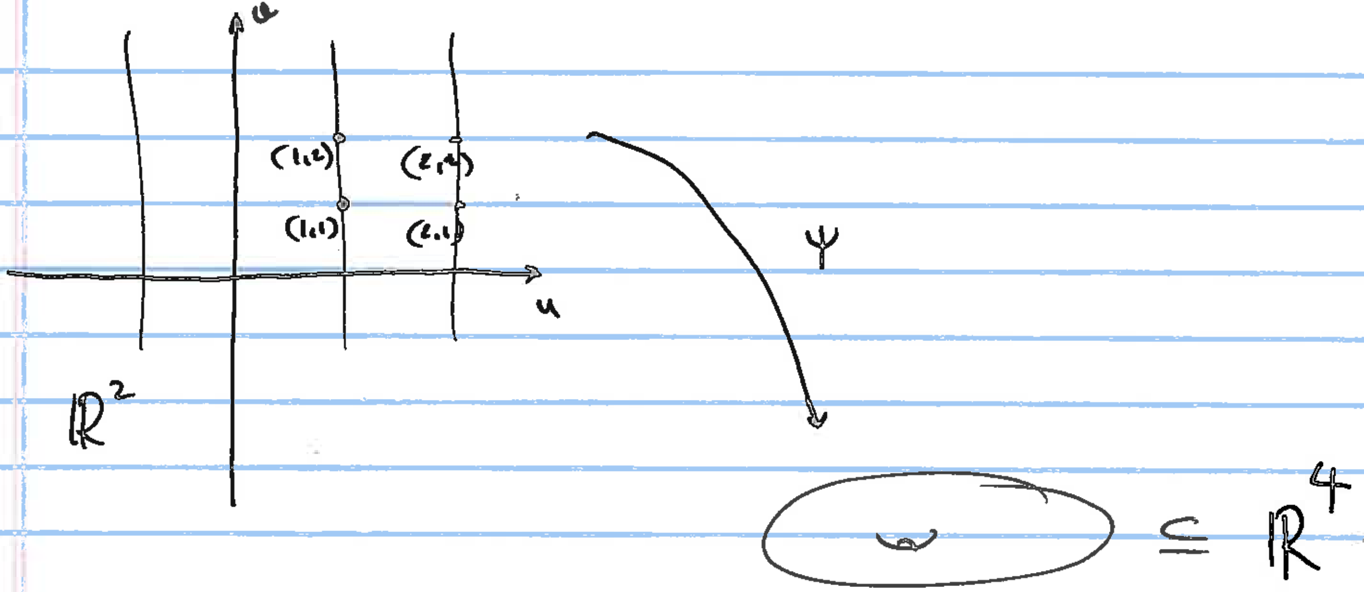
$$\Psi(u,v) = (\cos 2\pi u, \sin 2\pi u, \cos 2\pi v, \sin 2\pi v)$$

$$\Psi_u = (-2\pi \sin 2\pi u, 2\pi \cos 2\pi u, 0, 0)$$

$$\Psi_v = (0, 0, -2\pi \sin 2\pi v, 2\pi \cos 2\pi v$$

$$\left.\begin{aligned} E &= \Psi_u \cdot \Psi_u = 4\pi^2 \\ F &= \Psi_u \cdot \Psi_v = 0 \\ G &= \Psi_v \cdot \Psi_v = 4\pi^2 \end{aligned}\right\} \Rightarrow \Gamma^i_{jk} \equiv 0 \quad \forall i j k$$

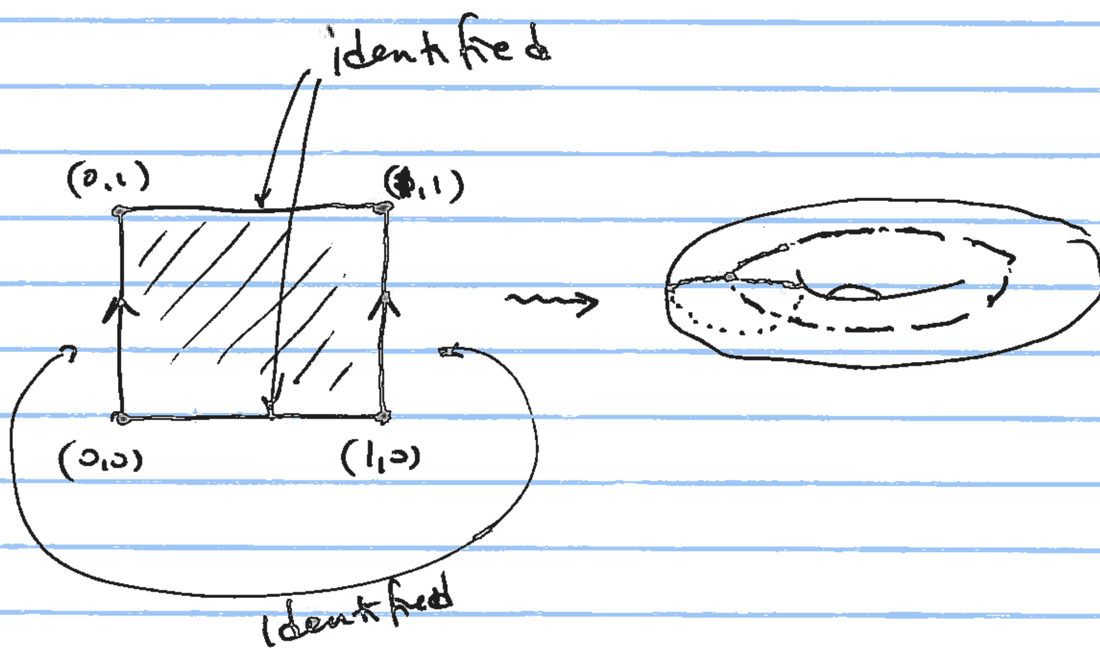
$$\Rightarrow K \equiv 0.$$



$$\Psi(u+p, v+q) = \Psi(u,v)$$

$\forall\, p,q \in \mathbb{Z}$ & $\Psi(p,q) = \Psi(0,0)$

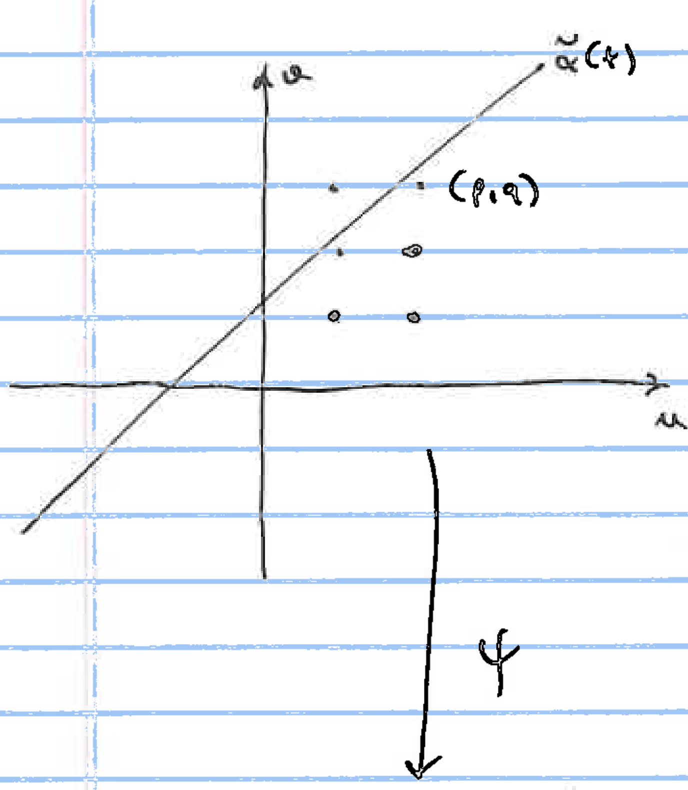
$\mathbb{Z} \times \mathbb{Z} = \{(p,q) \mid p,q \in \mathbb{Z}\}$ Integer Lattice



Geodesics of flat torus

$\forall\ \Gamma^i_{jk} \equiv 0$ Geodesic eq^{ns}. (in param. Domain)

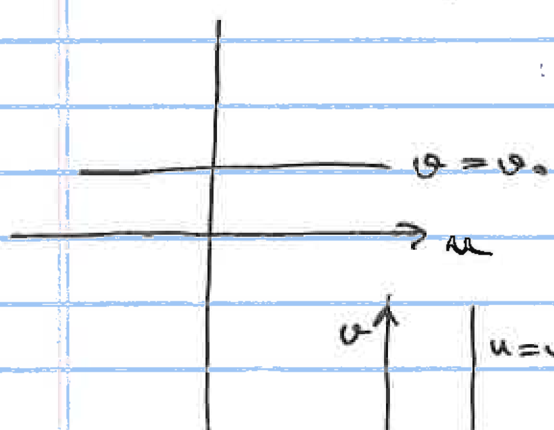
$$u'' = 0$$
$$v'' = 0$$
$$\Updownarrow$$
$$\tilde{\alpha}(t) \begin{cases} u = a + ct \\ v = b + dt \end{cases}$$



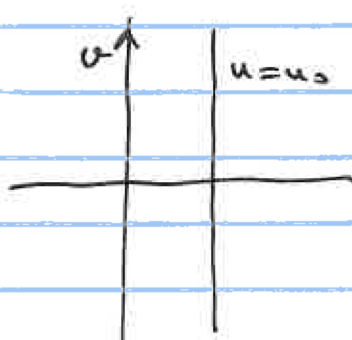
$\Psi(p,q) = \Psi(0,0)\ \forall p,q \in \mathbb{Z}$

$\Psi(\tilde{\alpha}(t)) = \alpha(t)$ geodesics of the flat torus in $\mathbb{R}^4$

What types of curves do I get?



$\subseteq \mathbb{R}^4$



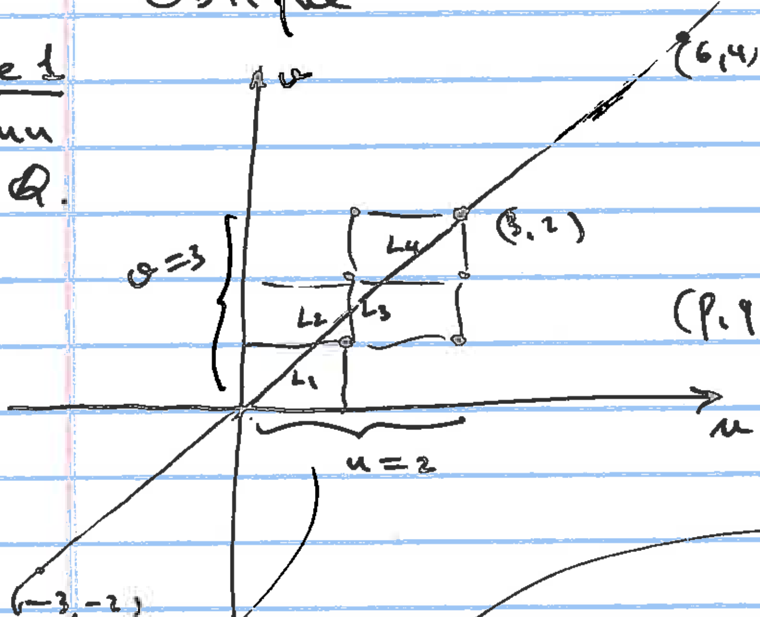
$\subseteq \mathbb{R}^4.$

Case 1

$v = mu$

$m \in \mathbb{Q}$.

Oblique

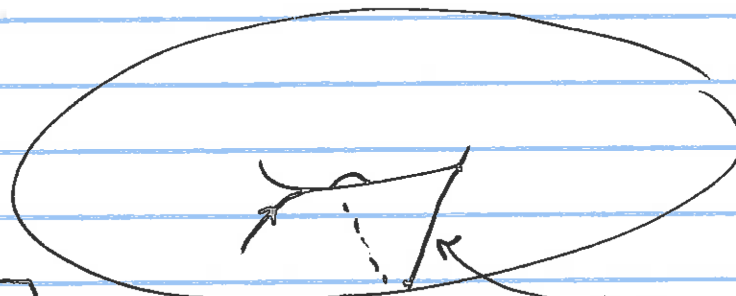


$v = \frac{3}{2} u$

$(p,q) \in \mathbb{Z} \times \mathbb{Z}$

$0 \le u \le p$.

$\psi(u, \underbrace{\tfrac{q}{p} u}_{v})$



wraps around q times

Closed geodesics

wraps around p times.

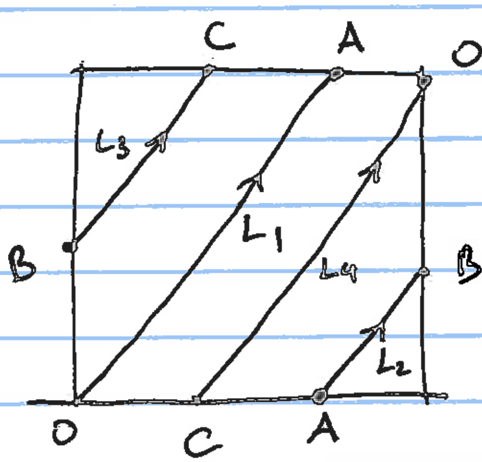
since $\psi(p,q) = \psi(0,0)$

$v = mu \qquad m \in \mathbb{Q}. \qquad m = \frac{q}{p}$

ψ

$p=2$

$q=3$



On torus / Top + Bottom edges identified left & right edges are identified

$0 \le u \le 2$

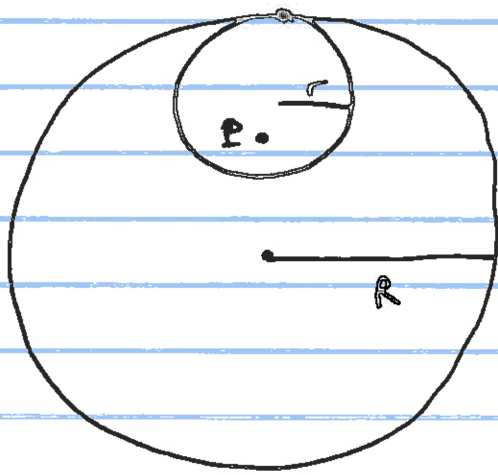
$\psi(u, \frac{3}{2} u)$ wraps around 2 times horizontally

wraps " 3 " vertically

Case 2 $\vartheta = mu$ $m \in \mathbb{R} - \mathbb{Q}$ irrational slope.

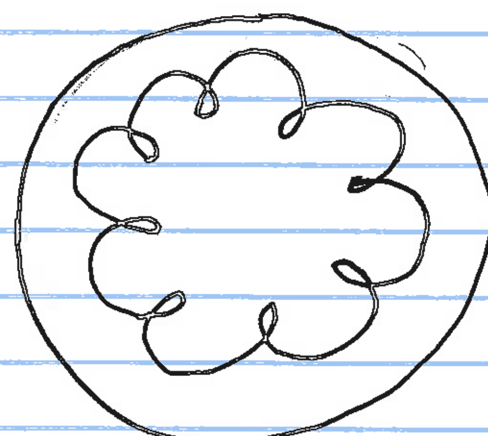
then $\Psi(u, mu)$ is a dense but not closed geodesic in $\mathbb{T}^2 \subseteq \mathbb{R}^4$.

Similar Behavior as in spirograph:



small circle rolls (without sliding) on the large circle, as we put a pencil at pt P, and trace a path.

If $\frac{R}{r} \in \mathbb{Q}$ then we get a closed curve.



If $\frac{R}{r} \in \mathbb{R} \setminus \mathbb{Q}$, then this curve never closes up, & we obtain a curve dense in an annular region: