5.1

<u>THM I</u> Let $\alpha(s) : [0, L] \to M$ be a C^∞ curve $|\alpha'| \equiv 1$. Suppose α is a shortest curve (locally in the path space), in M between $\alpha(0)$ and $\alpha(L)$.

Hence $\forall$ variation $F(s,r)$ s.t.

$$F(0,r) = \alpha(0)$$
$$F(L,r) = \alpha(L) \quad \forall r,$$

one has
$$\frac{d}{dr} \ell(\alpha_r(s)) \Big|_{r=} = 0.$$

Then $\underline{\alpha(s) \text{ is a geodesic}}$,

K_g of $\alpha \equiv 0$, and consequently

$$\alpha''_{tan}(s) \equiv 0$$

$$\Longrightarrow \quad \alpha''(s) \perp M \text{ at } \alpha(s)$$

i.e. $\alpha''(s) \perp T_{\alpha(s)} M$.

<u>Main idea</u> taking variations of type $\varphi(s)\,(U \times T)(s)$ near pts where $K_g(s_0) \neq 0$

bump. function about s_0.

then By using I^{st} V.F.

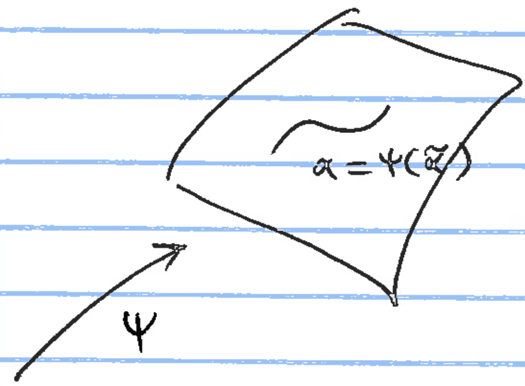
$$\frac{d}{dr} \ell(\alpha_r) \Big|_{r=0} < 0 \longrightarrow$$ when $K_g(s_0) \neq 0$ the curve α can't be shortest

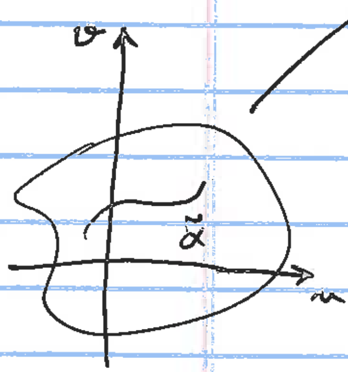
ODE's for geodesic

Thm II

Let M be a regular surface, Ψ be a local parametrization,

$$\alpha(t) = \Psi(\tilde{\alpha}(t)) = \Psi(\underbrace{u(t), v(t)}_{\tilde{\alpha}(t)})$$





Let

$$A(u,v) = u'' + \left(\Gamma^u_{uu} u'^2 + 2\Gamma^u_{uv} u'v' + \Gamma^u_{vv} v'^2\right.$$

$$B(u,v) = v'' + \left(\Gamma^v_{uu} u'^2 + 2\Gamma^u_{uv} u'v' + \Gamma^v_{vv} v'^2\right.$$

(Γ^i_{jk} associated with Ψ)

Then (i)

$$k_g \text{ of } \alpha = \sqrt{EG - F^2}\,(Bu' - Av')$$

(ii) Geodesic curvature is intrinsic

(iii) $k_g(\alpha) \equiv 0 \iff \alpha''_{tan} \equiv 0 \iff \underbrace{\begin{cases} A(u,v) = 0 \\ B(u,v) = 0 \end{cases}}_{\text{geodesic eq}^{ns}\text{ in the parametrization domain}}$ and

<u>Proof</u> : $\alpha(t) = \varphi(u(t), v(t))$

$$\alpha' = \varphi_u \cdot u' + \varphi_v \cdot v'$$

$$\alpha'' = \varphi_{uu} \cdot (u')^2 + \varphi_u \cdot u'' + \varphi_{uv}(v')(u') +$$

$$+ \varphi_{vu}(u')(v') + \varphi_{vv}(v')^2 + \varphi_v \cdot v''$$

Recall

$$\varphi_{uu} = \left(\Gamma^u_{uu} \varphi_u + \Gamma^v_{uu} \varphi_v + \ell U \right)$$

$$\varphi_{uv} = \left(\Gamma^u_{uv} \varphi_u + \Gamma^v_{uv} \varphi_v + m U \right)$$

$$\varphi_{vv} = \left(\Gamma^u_{vv} \varphi_u + \Gamma^v_{vv} \varphi_v + n U \right)$$

Plug in collect like terms

$$\alpha'' = \varphi_u \left[\Gamma^u_{uu} u'^2 + 2\Gamma^u_{uv} u'v' + \Gamma^u_{vv} v'^2 + u'' \right] +$$

$$+ \varphi_v \left[\Gamma^v_{uu} u'^2 + 2\Gamma^v_{uv} u'v' + \Gamma^v_{vv} v'^2 + v'' \right] +$$

$$+ U \left[\ell u'^2 + 2m u'v' + n v'^2 \right]$$

$$\alpha'' = \underbrace{A(u,v)\varphi_u + B(u,v)\varphi_v}_{\alpha''_{tan.}} + U\, C(u,v)$$

Recall α geodesic $\Longleftrightarrow$ $\alpha''_{\tan} = 0$

$$\Longleftrightarrow \left.\begin{array}{l} A = 0 \\ B = 0 \end{array}\right\}$$ parametrized domain geodesic Eq^n.

This proves (iii)

Thm III $\forall p \in M$ $\forall v \in T_pM$ $\exists\, \varepsilon > 0$

$\exists$ unique $\gamma(t) : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t.

(i) γ is a geodesic

(ii) $\gamma(0) = p$

(iii) $\gamma'(0) = v$

Proof FTODE + Prop II (iii)

Example 1

Any plane M in $\mathbb{R}^3$.

lines are geodesics:

$\alpha(t) = p_0 + tv_0$

$\alpha' = v_0$

$\alpha'' = 0 \implies \alpha''_{tan} \equiv 0.$

By Thm III $\forall p_0 \in M \ \forall v_0 \in T_{p_0}M$

$\exists !$ geodesic γ st. $\gamma(0) = p_0, \ \gamma'(0) = v_0$ but I know

$\alpha(t) = p_0 + v_0 t$

satisfies $\alpha(0) = p_0$

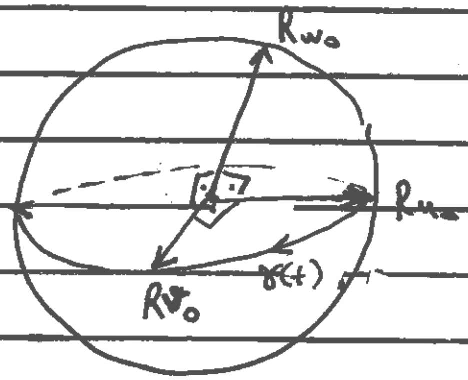
$\alpha'(0) = v_0 \implies \alpha = \gamma$

α geodesic.

We Got it all!

All lines of type $\alpha(t) = p_0 + tv_0$ are geodesics, and by Thm III, all geodesics of the plane are of that type.

Ex 2 S^2_R



Let u_0, v_0, w_0 be an o.n. basis of $\mathbb{R}^3$

$$\gamma(t) = R(\vec{u}_0 \cos t + \vec{v}_0 \sin t)$$

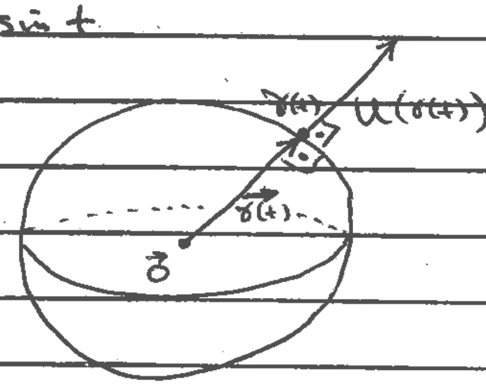
$$\gamma'(t) = -R\vec{u}_0 \sin t + R\vec{v}_0 \cos t$$

$$|\gamma'(t)| = R \quad \text{constant speed } (\Leftarrow \text{geodesic})$$

$$\gamma''(t) = -R\vec{u}_0 \cos t - R\vec{v}_0 \sin t$$

$$\gamma''(t) = -\gamma(t)$$

$$\gamma''_{\tan}(t) \equiv 0$$



On the sphere $\overrightarrow{\gamma(t)} \perp S^2$ at $\gamma(t)$

$$\overrightarrow{\gamma(t)} \perp T_{\gamma(t)} S^2$$

$$\overrightarrow{\gamma(t)} \parallel \bar{U}_{\gamma(t)}$$

$\gamma'' = -\gamma(t)$ has no tangential component

This is a great circle, i.e. intersection of S^2_R with the plane $\vec{x}\cdot(u_0 \times v_0) = 0 \leftarrow$ thru $\vec{0}$. (7)

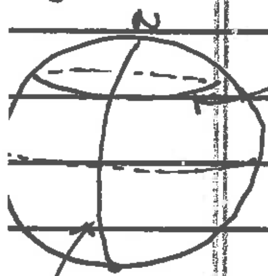
$$\underbrace{\gamma(t) = R\left(\vec{u}_0 \cos t + v_0 \sin t\right)}$$ is a geodesic

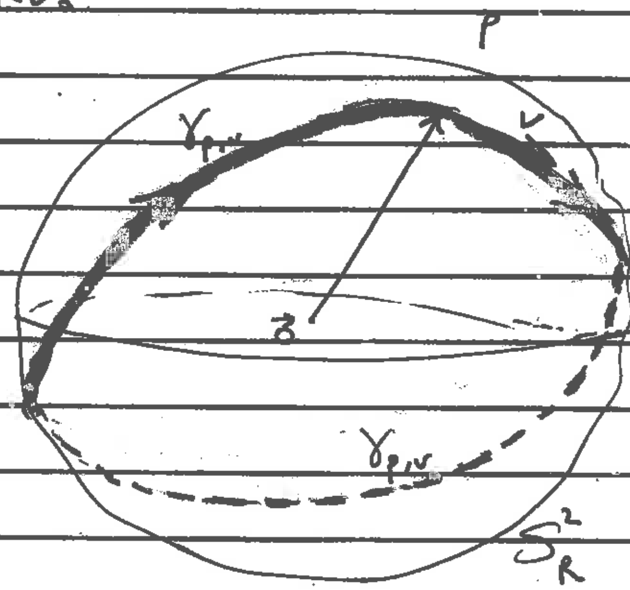
$\gamma(0) = R u_0$.

$\gamma'(0) = R v_0$

Given p, $v \in T_p M$
$\cap$
S^2_R

Caution not great circles are not geodesics





If $v \neq 0$ $\quad \gamma(t) = \vec{p}\cdot\cos t + \frac{\vec{v}}{|v|} R\cdot \sin t$ is the

geodesic with $\gamma(0) = p$
$\gamma'(0) = \frac{v}{|v|}\cdot R$.

If we want $\gamma'(0) = v$, we adjust the speed.

$$\gamma_{p,v} = \vec{p}\cos\left(t\cdot\frac{|v|}{R}\right) + \frac{\vec{v}}{|v|} R \sin\left(t\cdot\frac{|v|}{R}\right)$$

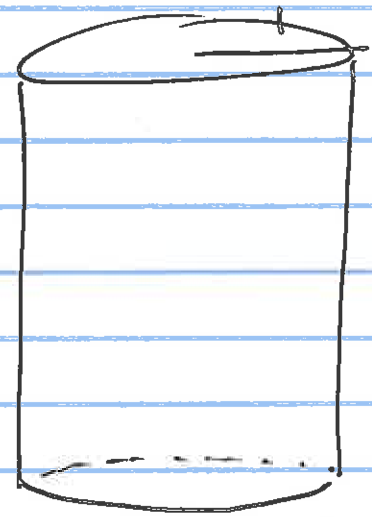
then $\gamma_{p,v}(0) = p$

$\gamma'_{p,v}(0) = \frac{\vec{v}}{|v|} R\cdot\frac{|v|}{R} = \vec{v}$

If $v = 0$, then $\gamma(t) = p$ pt curve is the geodesic.

Geodesics of
C : Cylinder

Ex



$\psi(u,v) = (\cos u, \sin u, v)$

$E = 1$
$F = 0$
$G = 1.$

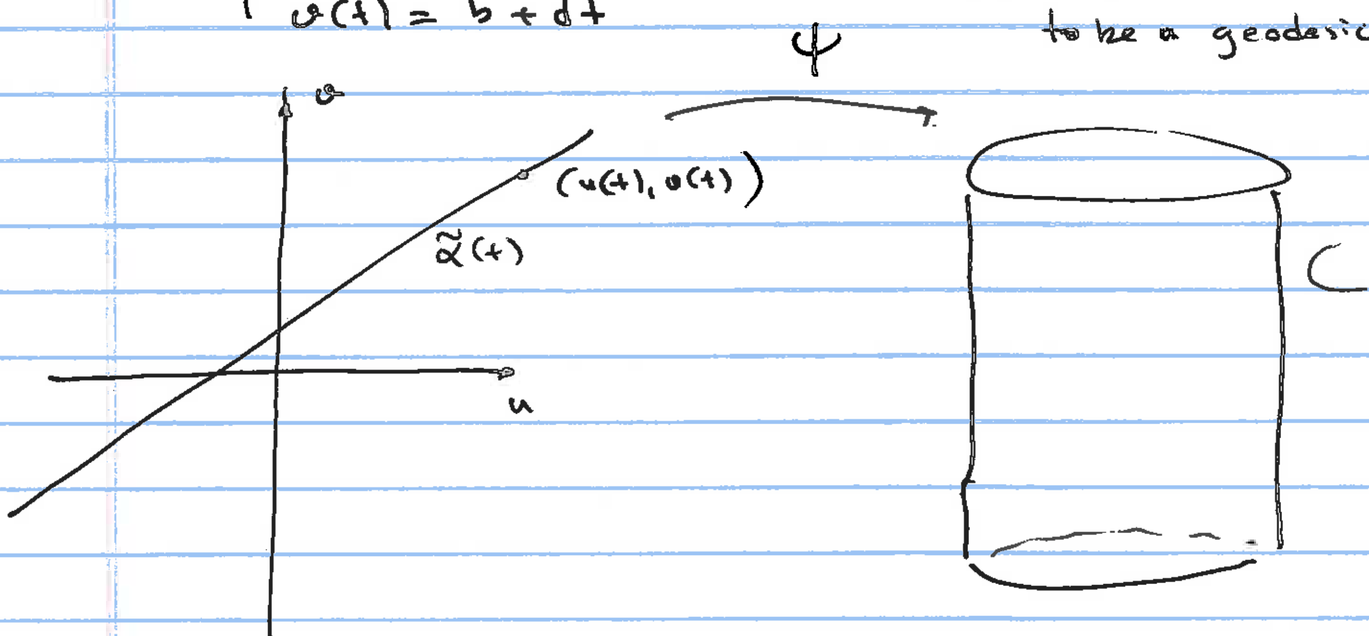
$\Longrightarrow \Gamma^i_{jk} \equiv 0 \quad \forall ijk$

involves partial deriv of E, F, G

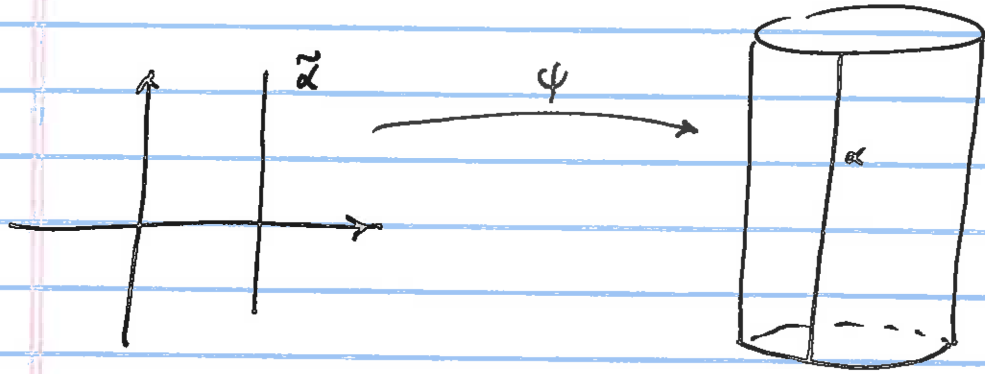
$$\left.\begin{array}{l} A(u,v) = u'' = 0 \\ B(u,v) = v'' = 0 \end{array}\right\} \text{geodesic eqn in the parametrization domain.}$$

$$(u(t), v(t)) = \tilde{\alpha}(t) \begin{cases} u(t) = a + ct \\ v(t) = b + dt \end{cases}$$

Want
$\psi(\tilde{\alpha}(t)) = \alpha(t)$
to be a geodesic of C

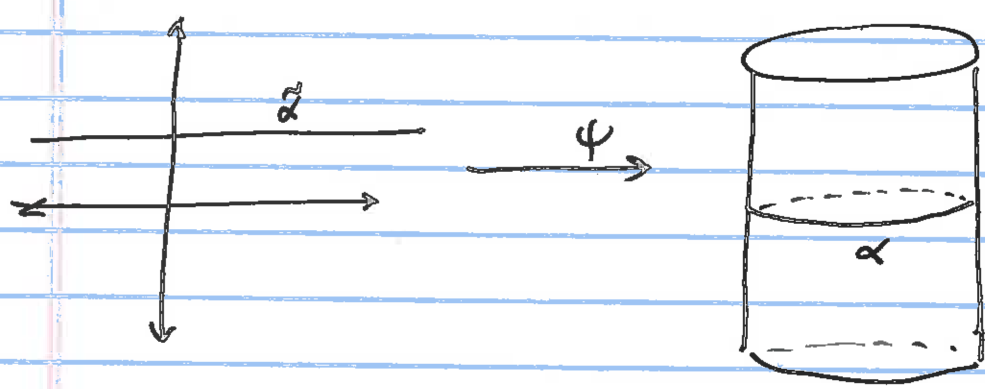


- If $\tilde{\alpha} = (u(t), v(t))$ is a vertical line in $\mathbb{R}^2$, one gets vertical line on C



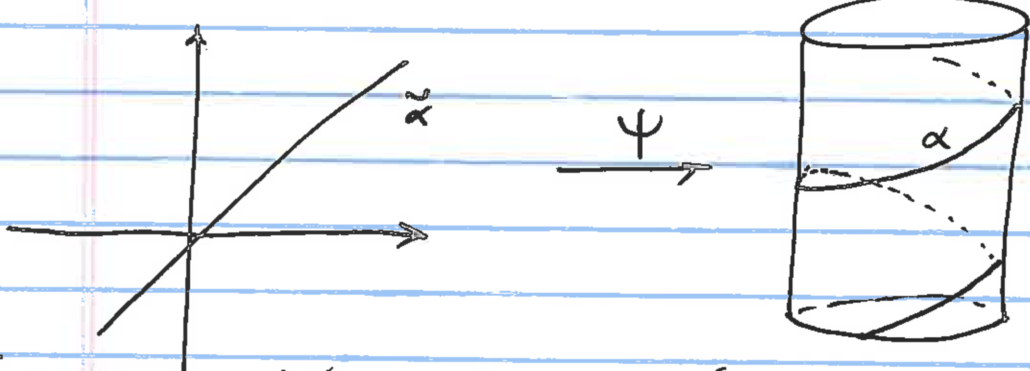
α is a geodesic of C

- If $(u(t), v(t))$ is a horizontal line, then one gets a circle on C



α is a Geodesic of C

If $(u(t), v(t))$ is an oblique line, then one gets a helix on C (a geodesic of C)



α is a Geodesic of C

Thm III

$\Rightarrow$ All Geodesics of C: $\psi(a+ct, b+dt) = (\cos(a+ct), \sin(a+ct), b+dt)$