Addition to Monday's Lecture: ① $\cdot$ $\kappa^2 = k_n^2 + k_g^2$, since

$|\alpha'| = 1 \Rightarrow \alpha'' = k_g\, U \times T + k_n\, U$ unit & $\perp$.

② Changing speed doesn't change κ, κ_g, k_n.

However:

(i) changing U orientation of the surface, and

(ii) changing the direction of the curve

may change the signs k_g, k_n

Thm: Leibnitz Rule

Let $f(s,t) : [a,b] \times [c,d] \xrightarrow{C^1} \mathbb{R}$, and

$$\int_a^b f(s,t)\,ds = g(t).$$

Then

$$g'(t) = \frac{d}{dt}\int_a^b f(s,t)\,ds = \int_a^b \frac{\partial}{\partial t} f(s,t)\,ds$$

$$= \int_a^b \frac{\partial f}{\partial t}(s,t)\,ds$$

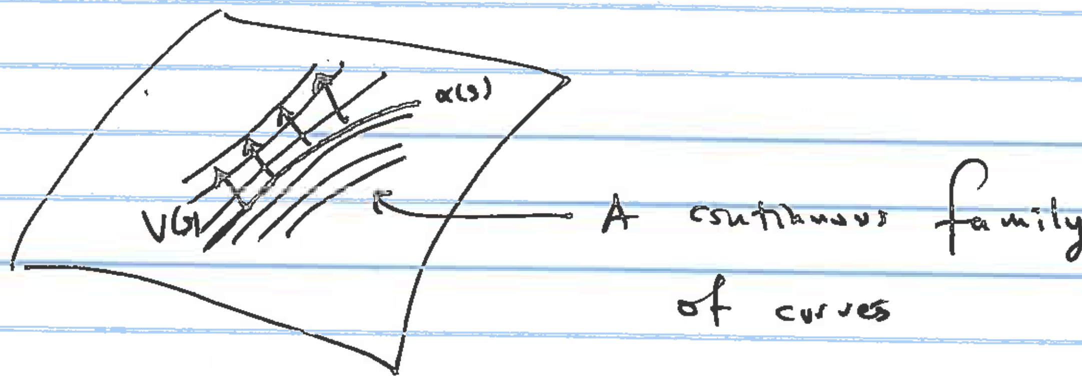
Remark: $\displaystyle\int_a^b \frac{\partial f}{\partial s}(s,t)\,ds = f(b,t) - f(a,t)$ FTC.

<u>Defn</u> Let $\alpha(s) : [0, L] \to M$ be a curve.
A function

$$F(s,r) : \overset{s}{[0,L]} \times \overset{r}{(-\varepsilon, \varepsilon)} \to M \quad \text{is}$$

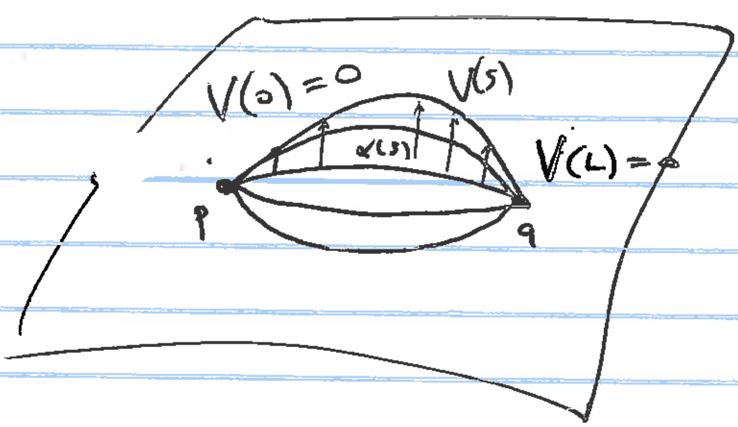
called a variation α if $F(s,0) = \alpha(s)$.

$V(s) = \dfrac{\partial F}{\partial r}(s,0)$ is called the variation vector field along α.



<u>Another example</u> end pts fixed

$$F(0,r) = p = \alpha(0)$$
$$F(L,r) = q = \alpha(L)$$
$$\forall r.$$



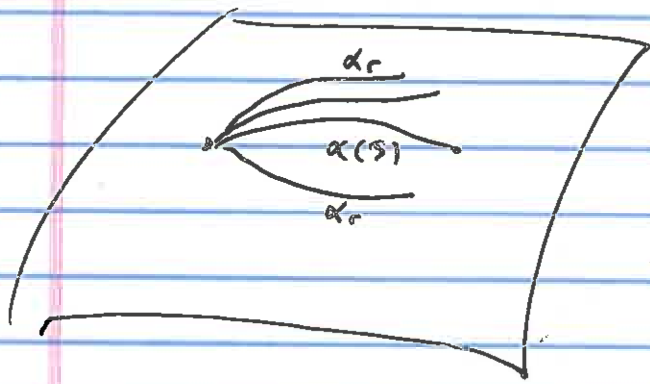
First Variation Formula:

Let M be a regular surface, $\alpha(s) : [0, L] \to M$ be a C^∞ curve, $|\alpha'(s)| \equiv 1$, Let $F(s,r)$ be a C^∞ variation of $\alpha(s)$.

We denote

$$\alpha_r(s) = F(s,r)$$

$$\ell(\alpha_r) = \text{length } \alpha_r(s)$$



Then: $$\frac{d}{dr} \ell(\alpha_r)\Big|_{r=0} = T(s)V(s)\Big|_0^L - \int_0^L T'(s)\, V(s)$$

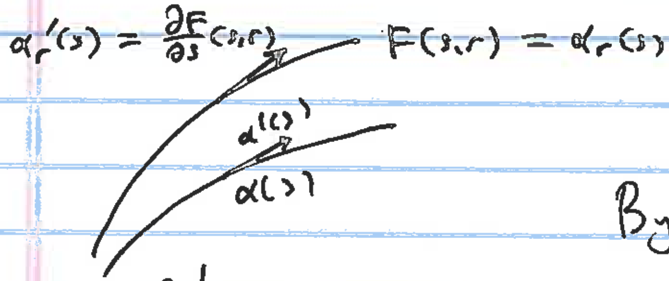
where $V(s) = \frac{\partial F}{\partial r}(s,0)$, $\quad T(s) = \alpha'(s)$.

$\alpha_r(s) = F(s,r)$

$\alpha_r'(s) = F_s(s,r)$

Proof 1^{st} VF

$$\frac{d}{dr}\left(\ell(\alpha_r)\right)\Big|_{r=0} = \frac{d}{dr}\int_0^L \left\|\frac{d}{ds}F(s,r)\right\| ds \,\Big|_{r=0}$$



By Leibnitz Rule

$$= \int_0^L \frac{d}{dr}\left\|\frac{\partial F}{\partial s}(s,r)\right\|\Big|_{r=0} ds$$

$$= \int_0^L \frac{d}{dr}\Big(\underbrace{\frac{\partial F}{\partial s}}_{F_s}(s,r)\cdot\underbrace{\frac{\partial F}{\partial s}}_{F_s}(s,r)\Big)^{\frac{1}{2}}\Big|_{r=0} ds$$

$$= \int_0^L \frac{d}{dr}(F_s\cdot F_s)^{\frac{1}{2}}\Big|_{r=0} ds = \int_0^L \frac{1}{2}(F_s\cdot F_s)^{-\frac{1}{2}}\cdot \underbrace{\frac{d}{dr}(F_s\cdot F_s)}\Big|_{r=0} ds$$

$$= \int_0^L \cancel{\tfrac{1}{2}}\underbrace{(F_s\cdot F_s)^{-\frac{1}{2}}}_{1 \text{ when } r=0}\cdot \cancel{2}F_{sr}\cdot F_s\Big|_{r=0} ds$$

$$= \int_0^L \underbrace{F_{rs}(s,0)}_{V'(s)}\cdot\underbrace{F_s(s,0)}_{\underbrace{\alpha'(s)}_{T}} ds$$

$F_s(s,0) = \alpha'(s)$

$|\alpha'(s)| = 1$

$F_{rs} = F_{sr}$

$V(s) = \frac{\partial F}{\partial r}(s,0)$

$V(s) = F_r(s,0)$

$$= \int_0^L V'(s)\cdot T(s)\, ds \overset{\text{By Parts}}{=} T(s)V(s)\Big|_0^L - \int_0^L T'(s)\,V(s)$$

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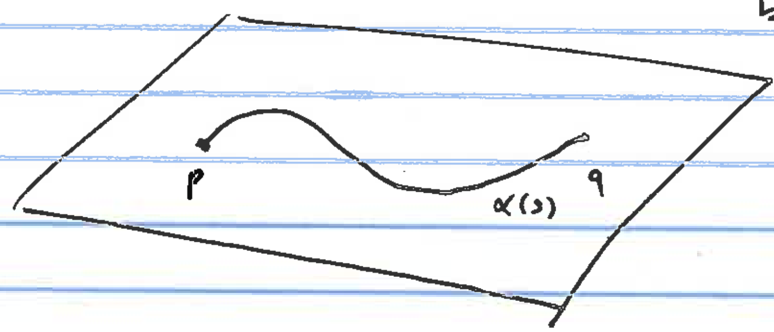
Example: $\mathbb{R}^2$ Why are the line segments shortest curves between its endpts?

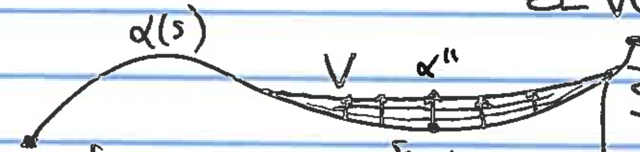


I^{st}VF: $\frac{d}{dr}(\ell(\alpha_r)) = T(s)V(s)\Big|_0^L - \int_0^L T'(s)V(s).$

Suppose α is not a line, $|\alpha'(s)| = 1$.

$|T'(s)| = |\alpha''(s)| = \kappa$, Suppose $\exists s_0$, $\alpha''(s_0) \neq 0$.
$\kappa(s_0) \neq 0$

$0 = V(L)$

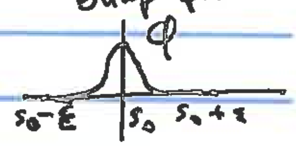


$\kappa \neq 0$, $\boxed{\text{Suppose } \alpha''(s_0) \neq 0}$

$V(0) = 0$

Let $V = (\alpha''(s)) \cdot \varphi$

small bump function



$\boxed{\text{A small perturbation of } \alpha \text{ in } \alpha'' \text{ direction, end pts fixed}} \to F(s,r) = \alpha(s) + rV$.

$$\frac{d}{dr}(\ell(\alpha_r)\Big|_{r=0} = \cancel{T(L)V(L) - T(0)V(0)}^{\,0} - \int_0^L T'(s)V(s)$$

$$= -\int_0^L \underbrace{\overbrace{\alpha''(s)}^{T'} \cdot \overbrace{\alpha''(s)}^{V}}_{\substack{\geq 0 \\ \text{about } s_0}} \underbrace{\varphi}_{\substack{> 0 \\ \text{about } s_0}}$$

$\varphi \equiv 0$ outside $[s_0 - \varepsilon, s_0 + \varepsilon]$

$$\frac{d}{dr}\ell(\alpha_r)\Big|_{r=0} = -\int_{s_0-\varepsilon}^{s_0+\varepsilon} \|\alpha''(s)\|^2 \varphi(s) < 0.$$

α can be shortened if $\alpha''(s_0) \neq 0$.